CURRENT BETWEEN TWO PLATES WITH A SPHERICAL PERTURBATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.38.

Another example of calculating the current flowing between two conductors separated by a weakly conducting medium. We have two large parallel plates separated by a distance d, with the lower plate at potential 0 and the upper plate at potential $V = V_0$. A metal hemisphere of radius $a \ll d$ rests on the lower plate so its potential is also zero. The region between the plates is filled with a substance of conductivity σ. We want the current flowing to the hemisphere.

The relation between current and electric field is

$$\mathbf{J} = \sigma\mathbf{E} \tag{1}$$

so we need the field around the hemisphere. This is worked out (well, the potential is anyway) for a complete sphere by using a series solution of Laplace's equation in Griffiths's example 3.8, so we can quote the result here:

$$V = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \tag{2}$$

Here E_0 is the uniform electric field surrounding the hemisphere and since the hemisphere is small relative to the separation of the plates, $E_0 = V_0/d$. We can use $\mathbf{E} = -\nabla V$ to get the field in spherical coordinates:

$$\mathbf{E} = \frac{V_0}{d}\left(1 + 2\frac{a^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} + \frac{V_0}{d}\left(-1 + \frac{a^3}{r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}} \tag{3}$$

Since we're concerned only with the field between the plates, and the plates themselves provide the uniform background field, we can also apply this formula to the hemisphere.

To get the current flowing onto the hemisphere, we need to integrate this over the surface of the hemisphere. Since the normal to the surface is parallel to $\hat{\mathbf{r}}$ only the E_r component contributes and we get, taking $r = a$ since we're integrating over the surface of the hemisphere:

$$I = \sigma \int \mathbf{E} \cdot d\mathbf{a} \tag{4}$$

$$= 3\frac{V_0}{d} \int_0^{\pi/2} \cos\theta \, (2\pi a \sin\theta) \, (a d\theta) \tag{5}$$

$$= \frac{3\pi\sigma a^2 V_0}{d} \tag{6}$$

where in the second line, the spherical surface element is the horizontal circumference of the sphere at angle θ (which is $2\pi a \sin\theta$) multiplied by the arc length on the surface of an increment in θ (which is $a d\theta$).