COPPER PIPES IN A CONDUCTING MEDIUM

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.39.

Here we'll revisit the problem of two parallel copper pipes, except this time we'll embed the pipes in a weakly conducting medium with conductivity σ and find the current that flows between them per unit length. The geometry of the problem is the same as before. We have two infinite copper pipes, each of radius a. The axis of one pipe is on the line $x=-d$, $(y=0)$ and the other is on $x=+d$. The potential of the one on the left is held at $-V_0$ and the one on the right at $+V_0$. We found the potential of this system is

$$V=\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(x+b)^2+z^2}{(x-b)^2+z^2} \tag{1}$$

where

$$\lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}\frac{d}{a}} \tag{2}$$

$$b = \sqrt{d^2-a^2} \tag{3}$$

The dual pipe problem is transformed into a problem involving two parallel wires lying along the lines $x=\pm b$ and carrying linear charge densities $\pm\lambda$. We can use Gauss's law to integrate the electric field over a cylinder of unit length around the wire with charge density $-\lambda$ to get the current flowing into the wire:

$$I=\sigma\int\mathbf{E}\cdot d\mathbf{a}=-\frac{\sigma\lambda}{\epsilon_0}=-\frac{2\pi\sigma V_0}{\cosh^{-1}\frac{d}{a}}=-\frac{2\pi\sigma V_0}{\ln\left(\frac{d}{a}+\sqrt{\left(\frac{d}{a}\right)^2-1}\right)} \tag{4}$$

Gauss's law says that the surface integral of $\mathbf{E}$ is q/ϵ_0, where q is the charge enclosed by the surface. The minus sign indicates that the field points towards the wire, and thus in the opposite direction to the surface normal so the current flows towards the wire.

Incidentally, if we wanted to follow the same procedure as in the previous problem (and not use the parallel wire equivalent problem), we need to find the electric field and then integrate it over a surface enclosing one of the pipes. This is a *lot* harder, but here's how it goes. It's easier to use a cylindrical coordinate system with its axis equal to the axis of one of the pipes, so let's choose the z axis to be along the line $x = -d$. We can then transform coordinates to $\xi \equiv x + d$ so the potential becomes

$$V = \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{(\xi + b - d)^2 + z^2}{(\xi - b - d)^2 + z^2} \tag{5}$$

In terms of the cylindrical coordinates, we have $\xi = r\cos\theta$, so

$$V = \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{r^2 + 2(b-d)\,r\cos\theta + (b-d)^2}{r^2 - 2(b+d)\,r\cos\theta + (b+d)^2} \tag{6}$$

Now to find the electric field we take the negative gradient, but since we're integrating over the surface of the pipe, the integrand $\mathbf{E}\cdot d\mathbf{a}$ will involve only the radial component $E_r = -\partial V/\partial r$. As this involves the logarithm of a quotient of two polynomials, the derivative isn't exactly simple and at this stage it's easier to turn the problem over to Maple. The derivative is too complex to write down here, but we can then integrate this over a cylinder of radius a (so that $r = a$ and the integral is over θ from 0 to 2π), and Maple confirms we get the same answer as above.