RESISTANCE IN A CONICAL CAPACITOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.40.

Here's another example of calculating the resistance. We have a truncated cone with the radius of the shorter end cap being a and of the longer being b, and the length between the two ends being L. The b end is held at constant potential V_0 and the a end is at $V=0$. What is the current between the ends if the cone is filled with a material with resistivity ρ?

If we make the (incorrect) assumption that the electric field between the ends is constant and parallel to the axis, we can derive the resistance as follows. Consider a thin slice of thickness dz through the cone. The voltage drop across this slice is $dV=V_0dz/L$, so the current flowing through the slice is

$$I=\frac{dV}{dz}\frac{1}{\rho}\pi r^2 \tag{1}$$

where r is the radius of the cone at point z. From Ohm's law, the resistance dR of the slice can be found:

$$\begin{aligned} dV &= IdR & (2)\\ dR &= \frac{\rho dz}{\pi r^2} & (3)\end{aligned}$$

We need r as a function of z which we can get from similar triangles. Suppose we extend the cone at the a end so that its vertex is restored, and call the distance from the vertex to the a circle's centre z_0. Then we get

$$\frac{a}{z_0}=\frac{r}{z+z_0}=\frac{b}{z_0+L} \tag{4}$$

where z is the distance along the axis from the a end towards to b end (so $z=0$ at a and $z=L$ at b).

We know everything in these equations except r and z_0, so we can solve to get them:

$$z_0 = \frac{aL}{b-a} \tag{5}$$

$$r = a\left(1+\frac{z}{z_0}\right) \tag{6}$$

$$= a\left(1+\frac{z(b-a)}{aL}\right) \tag{7}$$

We can now get the total resistance of the truncated cone as

$$R = \frac{\rho}{\pi}\int_0^L \frac{dz}{r^2} \tag{8}$$

$$= \frac{\rho}{\pi a^2}\int_0^L \left(1+\frac{z(b-a)}{aL}\right)^{-2} dz \tag{9}$$

$$= \frac{\rho L}{\pi ab} \tag{10}$$

The problem with this derivation is that **E** is not constant along the cone's length, since the field spreads out from the short end to the long end.

An alternative situation that *does* work out properly is obtained by replacing the flat ends of the cone with circular sections from a sphere centred at the cone's vertex. In this case the a end is a concave surface of radius a and the b end is a convex surface of radius b. We can take L to be the distance between the centres of the circles at either end of the cone. The parameters r and z_0 are therefore the same as before. This time the infinitesimal slices are concentric with the spherical cap and have a thickness dy (where y is now the radial coordinate, since we've used r and R for other things already). The area of a shell of spherical radius y is

$$A = 2\pi y^2 \int_0^{\theta_0} \sin\theta d\theta = 2\pi y^2 (1-\cos\theta_0) \tag{11}$$

where θ_0 is the angle subtended by the radius of the shell. (Note that this formula becomes $4\pi y^2$ for $\theta_0 = \pi$ so it correctly gives the surface area of a sphere).

The angle θ_0 is the same for all shells and from the lower end of the cone we have

$$\cos\theta_0 = \frac{z_0}{\sqrt{z_0^2+a^2}} \tag{12}$$

The resistance of a spherical shell is then

$$dR = \frac{\rho dy}{A} \tag{13}$$

so we need to sum up all the shells. We can do this two ways: by integrating over y or over z. First, we integrate over y. To do this, we need the end points, which are the spherical radii for each end of the cone. These are

$$y_a = \sqrt{z_0^2 + a^2} \tag{14}$$

$$y_b = \sqrt{(z_0 + L)^2 + b^2} \tag{15}$$

The resistance is then

$$R = \frac{\rho}{2\pi (1 - \cos\theta_0)} \int_{y_a}^{y_b} \frac{dy}{y^2} \tag{16}$$

$$= \frac{\rho (b-a)^2}{2ab\pi \left(\sqrt{L^2 + (b-a)^2} - L \right)} \tag{17}$$

If we do the integral over z we need to transform the differential. For a general circle within the cone, we have

$$y = \sqrt{(z_0 + z)^2 + r^2} \tag{18}$$

$$= \sqrt{(z_0 + z)^2 + \left[\frac{a}{z_0} (z_0 + z) \right]^2} \tag{19}$$

$$= (z_0 + z) \sqrt{1 + a^2 / z_0^2} \tag{20}$$

$$dy = \sqrt{1 + a^2 / z_0^2} dz \tag{21}$$

$$= \frac{1}{L} \sqrt{L^2 + (b-a)^2} dz \tag{22}$$

The resistance integral is then

$$R = \frac{\rho}{2\pi\left(1-\cos\theta_0\right)}\frac{1}{L}\sqrt{L^2+\left(b-a\right)^2}\int_0^L \frac{dz}{\left(z_0+z\right)^2+\left[\frac{a}{z_0}\left(z_0+z\right)\right]^2} \tag{23}$$

$$= \frac{\rho\left(b-a\right)^2}{2ab\pi\left(\sqrt{L^2+\left(b-a\right)^2}-L\right)} \tag{24}$$

after doing all the substitutions.