

RESISTANCE IN A CONICAL CAPACITOR

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.40.

Here's another example of calculating the resistance. We have a truncated cone with the radius of the shorter end cap being a and of the longer being b , and the length between the two ends being L . The b end is held at constant potential V_0 and the a end is at $V = 0$. What is the current between the ends if the cone is filled with a material with resistivity ρ ?

If we make the (incorrect) assumption that the electric field between the ends is constant and parallel to the axis, we can derive the resistance as follows. Consider a thin slice of thickness dz through the cone. The voltage drop across this slice is $dV = V_0 dz/L$, so the current flowing through the slice is

$$(1) \quad I = \frac{dV}{dz} \frac{1}{\rho} \pi r^2$$

where r is the radius of the cone at point z . From Ohm's law, the resistance dR of the slice can be found:

$$(2) \quad dV = IdR$$

$$(3) \quad dR = \frac{\rho dz}{\pi r^2}$$

We need r as a function of z which we can get from similar triangles. Suppose we extend the cone at the a end so that its vertex is restored, and call the distance from the vertex to the a circle's centre z_0 . Then we get

$$(4) \quad \frac{a}{z_0} = \frac{r}{z + z_0} = \frac{b}{z_0 + L}$$

where z is the distance along the axis from the a end towards to b end (so $z = 0$ at a and $z = L$ at b).

We know everything in these equations except r and z_0 , so we can solve to get them:

$$(5) \quad z_0 = \frac{aL}{b-a}$$

$$(6) \quad r = a \left(1 + \frac{z}{z_0} \right)$$

$$(7) \quad = a \left(1 + \frac{z(b-a)}{aL} \right)$$

We can now get the total resistance of the truncated cone as

$$(8) \quad R = \frac{\rho}{\pi} \int_0^L \frac{dz}{r^2}$$

$$(9) \quad = \frac{\rho}{\pi a^2} \int_0^L \left(1 + \frac{z(b-a)}{aL} \right)^{-2} dz$$

$$(10) \quad = \frac{\rho L}{\pi ab}$$

The problem with this derivation is that \mathbf{E} is not constant along the cone's length, since the field spreads out from the short end to the long end.

An alternative situation that *does* work out properly is obtained by replacing the flat ends of the cone with circular sections from a sphere centred at the cone's vertex. In this case the a end is a concave surface of radius a and the b end is a convex surface of radius b . We can take L to be the distance between the centres of the circles at either end of the cone. The parameters r and z_0 are therefore the same as before. This time the infinitesimal slices are concentric with the spherical cap and have a thickness dy (where y is now the radial coordinate, since we've used r and R for other things already). The area of a shell of spherical radius y is

$$(11) \quad A = 2\pi y^2 \int_0^{\theta_0} \sin \theta d\theta = 2\pi y^2 (1 - \cos \theta_0)$$

where θ_0 is the angle subtended by the radius of the shell. (Note that this formula becomes $4\pi y^2$ for $\theta_0 = \pi$ so it correctly gives the surface area of a sphere).

The angle θ_0 is the same for all shells and from the lower end of the cone we have

$$(12) \quad \cos \theta_0 = \frac{z_0}{\sqrt{z_0^2 + a^2}}$$

The resistance of a spherical shell is then

$$(13) \quad dR = \frac{\rho dy}{A}$$

so we need to sum up all the shells. We can do this two ways: by integrating over y or over z . First, we integrate over y . To do this, we need the end points, which are the spherical radii for each end of the cone. These are

$$(14) \quad y_a = \sqrt{z_0^2 + a^2}$$

$$(15) \quad y_b = \sqrt{(z_0 + L)^2 + b^2}$$

The resistance is then

$$(16) \quad R = \frac{\rho}{2\pi(1 - \cos \theta_0)} \int_{y_a}^{y_b} \frac{dy}{y^2}$$

$$(17) \quad = \frac{\rho (b - a)^2}{2ab\pi \left(\sqrt{L^2 + (b - a)^2} - L \right)}$$

If we do the integral over z we need to transform the differential. For a general circle within the cone, we have

$$(18) \quad y = \sqrt{(z_0 + z)^2 + r^2}$$

$$(19) \quad = \sqrt{(z_0 + z)^2 + \left[\frac{a}{z_0} (z_0 + z) \right]^2}$$

$$(20) \quad = (z_0 + z) \sqrt{1 + a^2/z_0^2}$$

$$(21) \quad dy = \sqrt{1 + a^2/z_0^2} dz$$

$$(22) \quad = \frac{1}{L} \sqrt{L^2 + (b - a)^2} dz$$

The resistance integral is then

$$(23) \quad R = \frac{\rho}{2\pi(1 - \cos \theta_0)} \frac{1}{L} \sqrt{L^2 + (b-a)^2} \int_0^L \frac{dz}{(z_0 + z)^2 + \left[\frac{a}{z_0} (z_0 + z) \right]^2}$$

$$(24) \quad = \frac{\rho (b-a)^2}{2ab\pi \left(\sqrt{L^2 + (b-a)^2} - L \right)}$$

after doing all the substitutions.