

## ELECTRIC POTENTIAL FROM A STEADY CURRENT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.41.

Mark A. Heald (1984), American Journal of Physics, **52**, 522.

Ohm's law is derived from the assumption that in a conducting medium  $\mathbf{J} = \sigma \mathbf{E}$ . If the currents are steady, then the charge density is independent of time, so  $\nabla \cdot \mathbf{J} = \frac{d\rho}{dt} = 0$ , which implies that (if  $\sigma$  is constant throughout the material)  $\nabla \cdot \mathbf{E} = -\nabla^2 V = 0$ , and we can use the various methods we derived earlier to solve Laplace's equation and determine the potential. One example of this is given here.

We have a cylindrical pipe of conductor of radius  $a$ , with a very thin slot cut down the side. A battery is attached to either side of the slot so that one edge of the slot is maintained at potential  $+V_0/2$  and the other at  $-V_0/2$ . This results in a uniform, steady current flowing around the circumference of the pipe. Since the conductivity of the pipe is uniform, the potential at an angle  $\phi$  on the surface of the pipe is

$$(0.1) \quad V(a, \phi) = \frac{V_0}{2\pi} \phi$$

for  $-\pi < \phi < +\pi$ . Our job is to find the potential both inside and outside the cylinder.

We can approach this problem using the series solution to Laplace's equation in cylindrical coordinates that we worked out earlier:

$$(0.2) \quad V(r, \phi) = A \ln r + B + \sum_{n=1}^{\infty} r^n (A_n \sin n\phi + B_n \cos n\phi) + \sum_{n=1}^{\infty} \frac{1}{r^n} (C_n \sin n\phi + D_n \cos n\phi)$$

with the boundary condition given above.

Inside the cylinder, to keep  $V$  finite we must throw away the log term and the series in inverse powers of  $r$ , so inside we have

$$(0.3) \quad V_{r < a} = B_i + \sum_{n=1}^{\infty} r^n (A_n \sin n\phi + B_n \cos n\phi)$$

Similarly, outside we get

$$(0.4) \quad V_{r>a} = B_o + \sum_{n=1}^{\infty} \frac{1}{r^n} (C_n \sin n\phi + D_n \cos n\phi)$$

At the boundary, we must have a continuous potential satisfying the boundary condition, so

$$(0.5) \quad B_i + \sum_{n=1}^{\infty} a^n (A_n \sin n\phi + B_n \cos n\phi) = \frac{V_0}{2\pi} \phi = B_o + \sum_{n=1}^{\infty} \frac{1}{a^n} (C_n \sin n\phi + D_n \cos n\phi)$$

At this point, we would like to be able to equate the coefficients of the sin and cos terms, but the boundary condition depends on  $\phi$  and not explicitly on sines or cosines. However, one of the nice things about Fourier series (which these series are, really) is that we can express any function in terms of them, so we can try to find a series such that

$$(0.6) \quad \frac{V_0}{2\pi} \phi = \sum_{n=1}^{\infty} F_n \sin n\phi$$

We can use the usual procedure for finding  $F_n$ : multiply both sides by  $\sin m\phi$  and integrate:

$$(0.7) \quad \frac{V_0}{2\pi} \int_{-\pi}^{\pi} \phi \sin m\phi d\phi = \sum_{n=1}^{\infty} F_n \int_{-\pi}^{\pi} \sin m\phi \sin n\phi d\phi$$

Since the sine function is orthogonal on the interval of integration, we get

$$(0.8) \quad \frac{V_0}{2\pi} \int_{-\pi}^{\pi} \phi \sin m\phi d\phi = F_m \int_{-\pi}^{\pi} \sin^2 m\phi d\phi$$

Working out the integrals, we get

$$(0.9) \quad \frac{(-1)^{m+1} V_0}{m} = \pi F_m$$

so

$$(0.10) \quad \frac{V_0}{2\pi} \phi = \frac{V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\phi$$

Now we can equate coefficients, so we get for  $r < a$

$$(0.11) \quad B_i = 0$$

$$(0.12) \quad B_n = 0$$

$$(0.13) \quad A_n = \frac{(-1)^{n+1}}{na^n}$$

$$(0.14) \quad V_{r < a} = -\frac{V_0}{\pi} \sum_{n=1}^{\infty} \left(-\frac{r}{a}\right)^n \frac{\sin n\phi}{n}$$

Similar reasoning on the outside gives

$$(0.15) \quad B_o = 0$$

$$(0.16) \quad D_n = 0$$

$$(0.17) \quad C_n = \frac{(-1)^{n+1} a^n}{n}$$

$$(0.18) \quad V_{r > a} = -\frac{V_0}{\pi} \sum_{n=1}^{\infty} \left(-\frac{a}{r}\right)^n \frac{\sin n\phi}{n}$$

To get the answer given in Griffiths, we need to sum the series. Maple is able to do this explicitly, and we get

$$(0.19) \quad V_{r > a} = \frac{V_0}{\pi} \arctan \left( \frac{\frac{a}{r} \sin \phi}{1 + \frac{a}{r} \cos \phi} \right)$$

$$(0.20) \quad = \frac{V_0}{\pi} \arctan \left( \frac{a \sin \phi}{r + a \cos \phi} \right)$$

The potential inside the cylinder is the same except with  $a$  and  $r$  swapped, so we have:

$$(0.21) \quad V_{r < a} = \frac{V_0}{\pi} \arctan \left( \frac{r \sin \phi}{a + r \cos \phi} \right)$$

We can check that these formulas do indeed satisfy the boundary condition using a couple of trig identities. Using double angle formulas we have

$$(0.22) \quad \sin \phi = 2 \sin \frac{\phi}{2} \cos \frac{\phi}{2}$$

$$(0.23) \quad \cos \phi = 2 \cos^2 \frac{\phi}{2} - 1$$

so at  $r = a$ , we have

$$(0.24) \quad V(a, \phi) = \frac{V_0}{\pi} \arctan \left( \frac{2a \sin \frac{\phi}{2} \cos \frac{\phi}{2}}{a \left( 1 + 2 \cos^2 \frac{\phi}{2} - 1 \right)} \right)$$

$$(0.25) \quad = \frac{V_0}{\pi} \arctan \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}}$$

$$(0.26) \quad = \frac{V_0}{2\pi} \phi$$

The surface charge density on the cylinder can be found from the radial component of the electric field at the surface, which is

$$(0.27) \quad E_r = - \left. \frac{\partial V}{\partial r} \right|_{r=a}$$

$$(0.28) \quad = \pm \frac{V_0 \sin \phi}{2\pi a (1 + \cos \phi)}$$

with the plus sign for the inner surface and the minus sign for the outer surface.

The surface charge density is  $\epsilon_0 E_r$ , or

$$(0.29) \quad \sigma = \pm \frac{\epsilon_0 V_0 \sin \phi}{2\pi a (1 + \cos \phi)}$$

(The answer given in Griffiths's question is clearly wrong; the charge density does not depend on  $r$  (what Griffiths calls  $s$ ), and this answer agrees with that given in Heald's paper.)