SURFACE CURRENT INDUCED BY A MAGNET FLOATING OVER A SUPERCONDUCTOR

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.43.

Here's another example of the effect of levitating a magnet above a superconductor. In the last post, we showed that the equilibrium position of the magnetic dipole occurs when it aligns itself horizontally above the planar superconductor. In this post, we'll look at what happens if the dipole is constrained to point normal to the plane. That is, we start with a dipole **m** which points in the $+z$ direction and sits at a height h above the superconducting plane.

As we saw in the previous post, the system can be modelled using the method of images, so the image dipole in this case is $-\mathbf{m}$ at position $z=-h$. Assuming the dipole has mass M, the height h can be found by equating the magnetic and gravitational forces. We've already found the formula for the force between two dipoles whose axes are parallel; to get the force when the axes are anti-parallel, we just reverse the sign, giving

$$\mathbf{F}=\frac{3\mu_0 m^2}{2\pi r^4}\hat{\mathbf{z}} \tag{1}$$

where r is the distance between the dipoles, so at equilibrium, $r=2h$. The height is thus found from

$$\frac{3\mu_0 m^2}{2\pi(2h)^4} = Mg \tag{2}$$

$$h = \frac{1}{2}\left(\frac{3\mu_0 m^2}{2\pi Mg}\right)^{1/4} \tag{3}$$

This is slightly higher (by a factor of $2^{1/4}$) than in the horizontal dipole case.

The above result was derived by using the fact that the normal component of **B** is continuous across boundaries. The tangential component of **B**, however, is not continuous; it depends on the surface current density **K**. The relation is

$$\mathbf{B}^{above} - \mathbf{B}^{below} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}} \tag{4}$$

In our case, $\mathbf{B}^{below} = 0$ due to the Meissner effect (exclusion of magnetic fields from a superconductor), so if we can find $\mathbf{B}^{above}$, we can get the surface current density. We start with the formula for the field due to the dipole:

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{m}\right] \tag{5}$$

Since the dipole points along the $+z$ direction, the system has cylindrical symmetry so consider a cross section lying in the xz plane, and choose a point on the x axis (which lies on the boundary with the superconductor). The unit vector $\hat{\mathbf{r}}$ points from the dipole to the point x and makes an angle θ with the z axis. Since $\mathbf{m} = m\hat{\mathbf{z}}$, $\mathbf{m}\cdot\hat{\mathbf{r}} = m\cos\theta$.

First, we show that there is no component of $\mathbf{B}$ in the y direction. The component $B_y = \mathbf{B}\cdot\hat{\mathbf{y}}$, but since $\hat{\mathbf{r}}$ lies in the xz plane $\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = 0$. Also, since $\mathbf{m} = m\hat{\mathbf{z}}$, $\mathbf{m}\cdot\hat{\mathbf{y}} = 0$, so $B_y = 0$.

Now to get $B_x = \mathbf{B}\cdot\hat{\mathbf{x}}$. The angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{x}}$ is $\theta - \frac{\pi}{2}$, so $\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} = \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$, and $\mathbf{m}\cdot\hat{\mathbf{x}} = 0$, so

$$B_x = \frac{\mu_0}{4\pi r^3}\left(3m\cos\theta\sin\theta\right) \tag{6}$$

By the way, the image solution does not, of course, give the correct answer for the field just inside the boundary (where the field is zero), but that's ok, since image solutions are valid only outside conductors.

To express the cos and sin in terms of the distances, consider the right triangle with vertices at the origin, the dipole and the point x. The vertical side has length h, the horizontal side has length x and the hypotenuse has length $r = \sqrt{h^2 + x^2}$. The angle between sides h and r is $\pi - \theta$ so the angle between sides x and r is $\pi - \left(\pi - \theta + \frac{\pi}{2}\right) = \theta - \frac{\pi}{2}$. Therefore

$$\frac{h}{r} = \sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta \tag{7}$$

$$\frac{x}{r} = \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta \tag{8}$$

Putting it all together, we get

$$B_x = -\frac{3\mu_0 mhx}{4\pi r^5} \tag{9}$$

$$= -\frac{3\mu_0 mhx}{4\pi\left(h^2 + x^2\right)^{5/2}} \tag{10}$$

By symmetry, the x component from the image dipole will be the same, so the total field is

$$B_{x;tot} = -\frac{3\mu_0 mhx}{2\pi\left(h^2+x^2\right)^{5/2}} \tag{11}$$

The minus sign indicates that the field points towards the origin, so the net field on the xy plane is radial, pointing inwards. That is, in cylindrical coordinates

$$\mathbf{B} = -\frac{3\mu_0 mhr}{2\pi\left(h^2+r^2\right)^{5/2}}\hat{\mathbf{r}} \tag{12}$$

where r now represents the cylindrical radius measured from the z axis.

The current density is therefore

$$\mathbf{K} = -\frac{3mhr}{2\pi\left(h^2+r^2\right)^{5/2}}\hat{\phi} \tag{13}$$

where here the minus sign indicates that the current flows in a clockwise direction when viewed from above.

It may seem odd that a static configuration of a magnet floating over a superconductor can give rise to a current, but remember that the magnet must be brought in to its final position and due to the Meissner effect, a surface current appears to negate the field inside the superconductor. Since the superconductor has zero resistance, once the current is established, it remains forever.