FLOATING A MAGNET ABOVE A SUPERCONDUCTOR

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.44.

A superconductor has a number of curious properties, one of which is the Meissner effect, which is the total exclusion of magnetic fields from the interior of the superconductor. Because the normal component of $\mathbf{B}$ is continuous across a boundary, $B_{\perp}=0$ on both sides of the boundary between the superconductor and the region above it.

Suppose we place magnet (which we'll idealize as an ideal magnetic dipole) above the superconductor, which is assumed to fill the half-space below the xy plane. We can apply the method of images to work out the field above the superconductor, and the resultant force on the dipole. We need an image dipole below the xy plane which, together with the 'real' dipole above, gives a zero normal component of B at the surface.

Let the z axis extend along a line that goes through the upper dipole and is normal to the xy plane. Suppose the upper dipole $\mathbf{m}_1$ is placed at a distance h above the plane and makes an angle θ with the z axis, and we place an image dipole $\mathbf{m}_2$ at a distance h below the plane, so that the angle between $\mathbf{m}_2$ and the z axis is $\pi-\theta$ (that is, $\mathbf{m}_2$ is an exact mirror image of $\mathbf{m}_1$, with $|\mathbf{m}_1|=|\mathbf{m}_2|\equiv m$). The fields due to the dipoles are

$$\mathbf{B}_i=\frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}_i\cdot\hat{\mathbf{r}}_i\right)\hat{\mathbf{r}}_i-\mathbf{m}_i\right] \tag{1}$$

where $i=1,2$. Now suppose we want the field at a point on the x axis, in the xy plane. The distance r from both dipoles to this point is the same. The normal (that is, z) component of the field is

$$\begin{aligned}
B_{iz} &= \mathbf{B}_i\cdot\hat{\mathbf{z}} && (2)\\
&= \frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}_i\cdot\hat{\mathbf{r}}_i\right)\left(\hat{\mathbf{r}}_i\cdot\hat{\mathbf{z}}\right)-\mathbf{m}_i\cdot\hat{\mathbf{z}}\right] && (3)
\end{aligned}$$

Because the two dipoles are mirror images, their z components are equal and opposite, but so are the z components of $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$. The result of this is that angle between $\mathbf{m}_1$ and $\hat{\mathbf{r}}_1$ is the same as the angle between $\mathbf{m}_2$ and $\hat{\mathbf{r}}_2$ so

$$\mathbf{m}_1\cdot\hat{\mathbf{r}}_1 = \mathbf{m}_2\cdot\hat{\mathbf{r}}_2 \tag{4}$$
$$\hat{\mathbf{r}}_1\cdot\hat{\mathbf{z}} = -\hat{\mathbf{r}}_2\cdot\hat{\mathbf{z}} \tag{5}$$
$$\mathbf{m}_1\cdot\hat{\mathbf{z}} = -\mathbf{m}_2\cdot\hat{\mathbf{z}} \tag{6}$$

Therefore, $B_{1z}=-B_{2z}$ so the total normal field is zero everywhere on the surface, no matter what the orientation of the upper dipole (provided its image is a true mirror image). Therefore, the condition on the normal field tells us nothing about the equilibrium orientation of the upper dipole. For that, we need the torque on a dipole in a field, which is

$$\mathbf{N}=\mathbf{m}\times\mathbf{B} \tag{7}$$

The torque on the upper dipole due to the image is thus

$$\mathbf{N} = \mathbf{m}_1\times\mathbf{B} \tag{8}$$
$$= \frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}_2\cdot\hat{\mathbf{r}}_2\right)\left(\mathbf{m}_1\times\hat{\mathbf{r}}_2\right)-\mathbf{m}_1\times\mathbf{m}_2\right] \tag{9}$$

Since both dipoles lie on the z axis, $\hat{\mathbf{r}}_2$ points in the $+z$ direction. The relevant angles are: between $\mathbf{m}_1$and $\hat{\mathbf{r}}_2$ the angle is θ; between $\mathbf{m}_2$ and $\hat{\mathbf{r}}_2$ the angle is $\pi-\theta$ and between $\mathbf{m}_1$ and $\mathbf{m}_2$ the angle is $\pi-\theta-\theta=\pi-2\theta$. To make things definite (and without loss of generality), we can take the two dipoles to lie in the xz plane, with positive x components, so that all cross products will lie along $\pm\hat{\mathbf{y}}$. We get for the various products of vectors:

$$\mathbf{m}_2\cdot\hat{\mathbf{r}}_2 = m\cos\left(\pi-\theta\right)=-m\cos\theta \tag{10}$$
$$\mathbf{m}_1\times\hat{\mathbf{r}}_2 = -m\sin\theta\hat{\mathbf{y}} \tag{11}$$
$$\mathbf{m}_1\times\mathbf{m}_2 = m^2\sin\left(\pi-2\theta\right)\hat{\mathbf{y}}=m^2\sin 2\theta\hat{\mathbf{y}}=2m^2\sin\theta\cos\theta\hat{\mathbf{y}} \tag{12}$$

Putting it all together, we get

$$\mathbf{N} = \frac{\mu_0}{4\pi r^3}\left[3m^2\sin\theta\cos\theta\hat{\mathbf{y}}-2m^2\sin\theta\cos\theta\hat{\mathbf{y}}\right] \tag{13}$$
$$= \frac{\mu_0}{4\pi r^3}m^2\sin\theta\cos\theta\hat{\mathbf{y}} \tag{14}$$
$$= \frac{\mu_0}{8\pi r^3}m^2\sin 2\theta\hat{\mathbf{y}} \tag{15}$$

The torque is therefore zero at $\theta=0,\frac{\pi}{2},\pi$, that is, when the upper dipole points straight up, straight down, or is horizontal (with any azimuthal angle). To see which of these angles is stable, note that if $0<\theta<\frac{\pi}{2}$, the torque is in the $+y$ direction, which tends to rotate the dipole *towards* the horizontal. Similarly, if $\frac{\pi}{2} < \theta < \pi$, the torque is in the $-y$ direction, which again rotates the dipole back towards the horizontal. Thus $\theta = \frac{\pi}{2}$ is the stable position, and the magnet will orient itself so that it is horizontal.

What force does the magnet feel? The formula for the force on a magnetic dipole is

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{16}$$

With both dipoles horizontal, we have for the field due to the image:

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_2 \cdot \hat{\mathbf{r}}_2)\hat{\mathbf{r}}_2 - \mathbf{m}_2] \tag{17}$$

This time, $\mathbf{m}_2$ is fixed while $\hat{\mathbf{r}}_2$ varies so to avoid confusion, we'll define the angle between $\hat{\mathbf{r}}_2$ and the $+z$ axis to be α. Since $\mathbf{m}_2$ is horizontal, the angle between $\mathbf{m}_2$ and $\hat{\mathbf{r}}_2$ is $\frac{\pi}{2} - \alpha$ so

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0}{4\pi r^3}\left[3m\cos\left(\frac{\pi}{2} - \alpha\right)\hat{\mathbf{r}}_2 - \mathbf{m}_2\right] && (18) \\ &= \frac{\mu_0}{4\pi r^3}[3m\sin\alpha\hat{\mathbf{r}}_2 - \mathbf{m}_2] && (19) \end{aligned}$$

Since $\mathbf{m}_1$ and $\mathbf{m}_2$ are parallel (they are actually the same vector), we then get

$$\begin{aligned} \mathbf{m}_1 \cdot \mathbf{B} &= \frac{\mu_0}{4\pi r^3}\left[3m^2\sin^2\alpha - m^2\right] && (20) \\ &= \frac{\mu_0 m^2}{4\pi r^3}\left[3\sin^2\alpha - 1\right] && (21) \end{aligned}$$

We can now revert to the usual spherical coordinates (that is, we'll replace α by θ_2, where the subscript 2 reminds us that the centre of the spherical coordinate system is the image dipole) to get

$$\begin{aligned} \mathbf{F} &= \nabla(\mathbf{m}_1 \cdot \mathbf{B}) && (22) \\ &= \nabla\left[\frac{\mu_0 m^2}{4\pi r^3}\left[3\sin^2\theta_2 - 1\right]\right] && (23) \\ &= -\frac{3\mu_0 m^2}{4\pi r^4}\left[3\sin^2\theta_2 - 1\right]\hat{\mathbf{r}}_2 + \frac{6\mu_0 m^2}{4\pi r^4}\sin\theta_2\cos\theta_2\hat{\theta}_2 && (24) \end{aligned}$$

We're interested in the force when the dipole is at $\theta_2 = 0$, where we get

$$\mathbf{F}(\theta_2 = 0) = \frac{3\mu_0 m^2}{4\pi r^4}\hat{\mathbf{r}}_2 \tag{25}$$

Since this points in the $+z$ direction, the force is repulsive, meaning that the magnet floats above the superconductor. To get the height, we can give the magnet a mass of M and equate the magnetic force with the gravitational force:

$$\frac{3\mu_0 m^2}{4\pi r^4} = Mg \tag{26}$$

$$r = \left(\frac{3\mu_0 m^2}{4\pi Mg}\right)^{1/4} \tag{27}$$

Since r is the distance from the image dipole, the distance above the xy plane is half that, or

$$h = \frac{1}{2}\left(\frac{3\mu_0 m^2}{4\pi Mg}\right)^{1/4} \tag{28}$$

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