DROPPING LOOPS THROUGH MAGNETIC FIELDS

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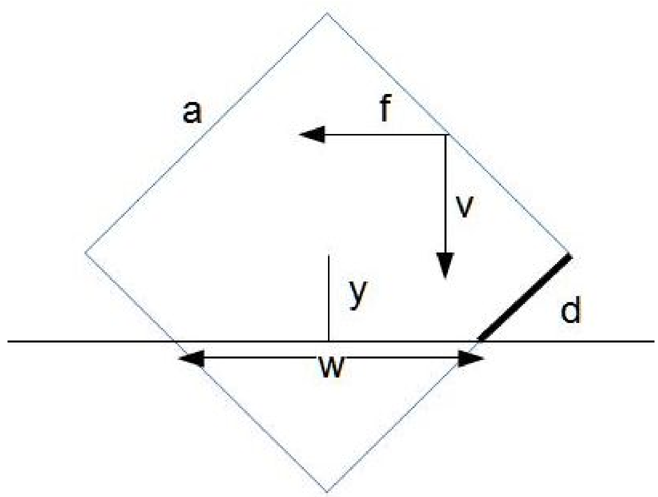
References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.46.

Here's a questionable variant on the problem of a metal loop falling through a magnetic field. We begin, as before, with a square loop of side length a, except this time we rotate it by $\pi/4$ so that it makes a diamond shape. In the diagram, the magnetic field points out of the page at all points above the horizontal line, and the loop moves downward under gravity, so the force per unit charge **f** points to the left, as shown in the figure:



If we now drop it from rest, what happens? As before, we first calculate the emf around the square, and take the path of integration as counterclockwise. For the top left side of the square, which is entirely within the magnetic field, we have

$$\mathcal{E}_{up-left} = \int \mathbf{f}\cdot d\ell = \frac{vBa}{\sqrt{2}} \tag{1}$$

By symmetry, we get the same contribution for the upper right, so the emf for the top two edges is

$$\mathcal{E}_{top} = \sqrt{2}vBa \tag{2}$$

The bottom two edges are only partially within the field, so we need to find the distance d. The ratio of d to a full side a is the same as the ratio of the height of the centre of the square y to half a diagonal. That is:

$$\frac{d}{a} = \frac{y}{a/\sqrt{2}} \tag{3}$$

$$d = \sqrt{2}y \tag{4}$$

The emf from the bottom two edges opposes that from the top two edges, so

$$\mathcal{E}_{bottom} = -2 \times \frac{vBd}{\sqrt{2}} = 2vBy \tag{5}$$

The total emf is thus

$$\mathcal{E} = vB\left(\sqrt{2}a - 2y\right) \tag{6}$$

If the resistance of the loop is R, this emf generates a current according to Ohm's law:

$$I = \frac{\mathcal{E}}{R} = \frac{vB}{R}\left(\sqrt{2}a - 2y\right) \tag{7}$$

We've seen that the force on a current loop that is partially enclosed in a magnetic field is

$$F = IwB \tag{8}$$

where w is the length of the chord at the edge of the field. Here, w is as shown in the diagram, and by similar triangles

$$\frac{w}{\sqrt{2}a} = \frac{\frac{a}{\sqrt{2}} - y}{\frac{a}{\sqrt{2}}} \tag{9}$$

$$w = \sqrt{2}a - 2y \tag{10}$$

Thus the force on the loop is

$$F = \frac{vB^2}{R}\left(\sqrt{2}a - 2y\right)^2 \tag{11}$$

If we follow the same argument here as we did for the square loop with its edges parallel to the edge of the field, we then say

$$\begin{aligned} m\dot{v} &= mg - F \qquad (12) \\ &= mg - \frac{vB^2}{R}\left(\sqrt{2}a - 2y\right)^2 \qquad (13) \end{aligned}$$

For the terminal velocity, we set $\dot{v} = 0$, so we get

$$v_{term2} = \frac{mgR}{B^2\left(\sqrt{2}a - 2y\right)^2} \qquad (14)$$

The original terminal velocity was

$$v_{term1} = \frac{mgR}{a^2B^2} \qquad (15)$$

so the ratio is

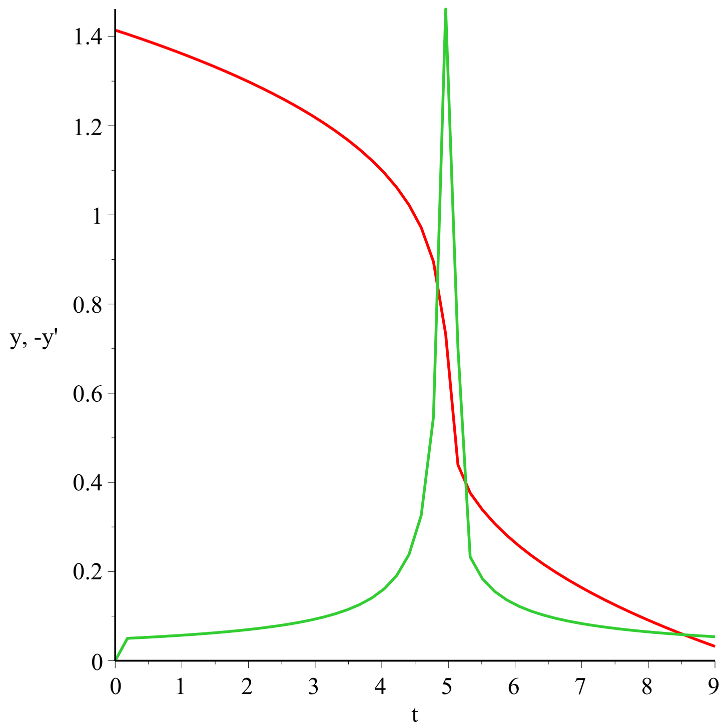
$$\frac{v_{term1}}{v_{term2}} = \left(\sqrt{2} - 2\frac{y}{a}\right)^2 \qquad (16)$$

which agrees with the answer in Griffiths. However, this derivation doesn't really make any sense, since the velocity $v = -\dot{y}$, so if $\dot{v} = 0$, this implies that v is a constant (presumably not zero). In that case, y is continually decreasing. However, 14 implies that if $v = v_{term2} = constant$, y must also be a constant, giving a contradiction.

I think to treat this properly, we need to write the equation of motion entirely in terms of y, giving

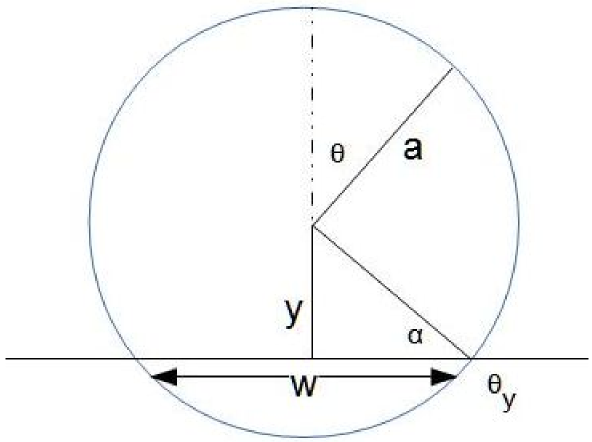
$$-m\ddot{y} = mg + \frac{B^2}{R}\dot{y}\left(\sqrt{2}a - 2y\right)^2 \qquad (17)$$

with initial conditions $y(0) = a/\sqrt{2}$ and $\dot{y}(0) = 0$. Not surprisingly, this non-linear ODE cannot be solved analytically, but using Maple, we can get a numerical solution, which is shown for $m = 1$, $g = 9.8$, $B = 1$, $a = 1$ and $R = 0.01$. The plot shows $y(t)$ (in red) and $\dot{y}(t)$ (in green).



This shows the portion where y decreases from $a/\sqrt{2}$ to zero (that is, as the lower half of the diamond falls across the boundary of the field).The spike in v around $t = 5$ is actually a smooth peak as can be seen with a more detailed graph. Also, the sharp corner in v just after $t = 0$ is also smooth. The latter looks a bit like the velocity rising rapidly and then levelling off and in fact, plugging the numbers into 14, we see that v at this point does actually level off (roughly) at a value close to v_{term2}. The spike in v from the solution of the ODE occurs around $y = a/\sqrt{2}$, at which point $v_{term2} \to \infty$, which is clearly nonsense, so the calculations here give only an approximation to the true behaviour.

Using similar approximate reasoning, we can work out what happens if we drop a circular loop of radius a in the same field. This time we'll let y be the vertical distance between the circle's centre and the edge of the field, and describe a point on the circle by an angle θ, taking $\theta = 0$ to be the point at the top of the circle.



The field covers the circle from $\theta = -\theta_y$ to $\theta = +\theta_y$, where $\theta_y = \frac{\pi}{2} + \alpha$, and $\alpha = \arcsin \frac{y}{a}$. The force **f** still points to the left and **v** points down, but now $d\ell$ is tangent to the circle, pointing counterclockwise. The angle between **f** and $d\ell$ is θ, so the emf is

$$
\begin{aligned}
\mathcal{E} &= \int_{-\theta_y}^{\theta_y} vB\cos\theta\,(a\,d\theta) && (18)\\
&= 2\int_{0}^{\theta_y} vB\cos\theta\,(a\,d\theta) && (19)\\
&= 2avB\sin\theta_y && (20)\\
&= 2avB\sin\left(\frac{\pi}{2} + \arcsin\frac{y}{a}\right) && (21)\\
&= 2avB\cos\left(\arcsin\frac{y}{a}\right) && (22)\\
&= 2avB\sqrt{1-\left(\frac{y}{a}\right)^2} && (23)
\end{aligned}
$$

From this we get the current:

$$I = \frac{\mathcal{E}}{R} = \frac{2avB}{R}\sqrt{1-\left(\frac{y}{a}\right)^2} \tag{24}$$

To get the magnetic force on the loop, we need the chord w, but this is

$$\frac{w}{2} = a\cos\alpha = a\sqrt{1-\left(\frac{y}{a}\right)^2} \tag{25}$$

The force is therefore

$$\begin{aligned} F &= IwB \qquad (26) \\ &= \frac{4a^2vB^2}{R}\left(1-\left(\frac{y}{a}\right)^2\right) \qquad (27) \end{aligned}$$

Again, taking $\dot{v}=0$ we get

$$\begin{aligned} \frac{4a^2vB^2}{R}\left(1-\left(\frac{y}{a}\right)^2\right) &= mg \qquad (28) \\ v &= \frac{mgR}{4a^2B^2}\left(1-\left(\frac{y}{a}\right)^2\right)^{-1} \qquad (29) \end{aligned}$$

The time taken for the loop to fall completely out of the field area is found from $v = dy/dt$

$$\begin{aligned} t &= \frac{4a^2B^2}{mgR}\int_a^{-a}\left(1-\left(\frac{y}{a}\right)^2\right)dy \qquad (30) \\ &= \frac{16a^3B^2}{3mgR} \qquad (31) \end{aligned}$$