

FARADAY FIELD AND MAGNETIC VECTOR POTENTIAL

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 5.50a, 7.47.

We've seen that in the magnetostatic case, the Biot-Savart law gives the magnetic field produced by steady currents:

$$(0.1) \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'$$

This law can be used to derive Ampère's law which in differential form is

$$(0.2) \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

The Biot-Savart law can be seen as the solution of Ampère's law, combined with the divergence equation $\nabla \cdot \mathbf{B} = 0$.

In the case where the magnetic field is changing with time, Faraday's law gives the electric field produced as a result of the changing magnetic field:

$$(0.3) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

If there is no free charge, then $\nabla \cdot \mathbf{E} = 0$, so this pair of equations for the electric field are formally equivalent to those above for the magnetic field. We can therefore write down an analogous solution by substituting $-\frac{\partial \mathbf{B}}{\partial t}$ for $\mu_0 \mathbf{J}$ in the Biot-Savart law to get

$$(0.4) \quad \mathbf{E} = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int_V \frac{\mathbf{B}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'$$

We can extend the same argument to the magnetic vector potential \mathbf{A} . We have $\mathbf{B} = \nabla \times \mathbf{A}$ and we can choose $\nabla \cdot \mathbf{A} = 0$, so again by analogy with the Biot-Savart law we can write

$$(0.5) \quad \mathbf{A} = \frac{1}{4\pi} \int_V \frac{\mathbf{B}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'$$

Comparing the last two equations, we get

$$(0.6) \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

Taking the curl of both sides gives us back Faraday's law: $\nabla \times \mathbf{E} = -\frac{\partial \nabla \times \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{B}}{\partial t}$. Remember that this is true *only* for those electric fields produced as a result of changing magnetic fields; it does *not* apply to electric fields produced by free charge.

Suppose now we have a spherical shell of radius R with a uniform surface charge density σ which spins at a variable rate $\omega(t)$. Griffiths works out the vector potential of this sphere in his example 5.11 and gets

$$(0.7) \quad \mathbf{A} = \begin{cases} \frac{\mu_0 R \omega \sigma}{3} r \sin \theta \hat{\phi} & r \leq R \\ \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin \theta}{r^2} \hat{\phi} & r \geq R \end{cases}$$

Inside the sphere there is no Coulomb field, while outside, this field is

$$(0.8) \quad \mathbf{E}_{coulomb} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

$$(0.9) \quad = \frac{4\pi R^2 \sigma}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

$$(0.10) \quad = \frac{R^2 \sigma}{\epsilon_0 r^2} \hat{\mathbf{r}}$$

Using 0.6 for the Faraday field, we get the total field:

$$(0.11) \quad \mathbf{E} = \begin{cases} -\frac{\mu_0 R \dot{\omega} \sigma}{3} r \sin \theta \hat{\phi} & r \leq R \\ \frac{R^2 \sigma}{\epsilon_0 r^2} \hat{\mathbf{r}} - \frac{\mu_0 R^4 \dot{\omega} \sigma}{3} \frac{\sin \theta}{r^2} \hat{\phi} & r \geq R \end{cases}$$