ACCELERATING AN ATOMIC ELECTRON WITH A CHANGING MAGNETIC FIELD

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.49.

This is another example of the cyclotron field, though this time we're dealing with an electron orbiting an atomic nucleus. The treatment here is purely classical, so it's certainly not a valid model of how a magnetic field interacts with an atom (for that we need quantum mechanics). However, let's have a look and see where the argument takes us.

Suppose an electron of charge $-q$ orbits a nucleus of charge Q, and initially there are no external magnetic or electric fields. Equating centripetal and Coulomb forces, we get

$$\frac{mv^2}{r}=\frac{Qq}{4\pi\epsilon_0 r^2} \tag{1}$$

Now suppose we turn on a magnetic field perpendicular to the plane of the orbit, in the $+z$ direction. In time dt the field increases by an amount dB, so that the force equation now becomes (assuming r doesn't change, which we'll justify below):

$$\frac{mv_1^2}{r}=\frac{Qq}{4\pi\epsilon_0 r^2}-qv_1 dB \tag{2}$$

where v_1 is the new velocity of the electron. The minus sign occurs because we're taking the direction of the orbit to be the $+\phi$ direction (clockwise as viewed from above). This results in a force pointing radially outwards (for positive q), so it detracts from the attractive Coulomb force. Griffiths works out the difference v_1-v in his section 6.1.3, but we won't need the actual value here; rather we'll just say that v changes by a small amount so that $v_1-v=dv$. To first order:

$$\frac{mv_1^2}{r} = \frac{mv^2}{r} + 2\frac{mv}{r}dv \tag{3}$$
$$= \frac{Qq}{4\pi\epsilon_0 r^2} - qv\,dB \tag{4}$$
$$= \frac{mv^2}{r} - qv\,dB \tag{5}$$

The change in kinetic energy due to the change in v is therefore

$$dT = \frac{1}{2}mv_1^2 - \frac{1}{2}mv^2 \tag{6}$$
$$= -\frac{1}{2}qrv\,dB \tag{7}$$

Since magnetic forces always act perpendicular to the direction of motion, they never do any work, so this change in kinetic energy must come from somewhere else. Since a changing magnetic field gives rise to an electric field by Faraday's law, and electric forces *can* do work, that must be the source of the change in energy. If we can show that this induced electric field gives exactly this energy, then we've justified the assumption that r doesn't change. Faraday's law in integral form says

$$\oint \mathbf{E}\cdot d\ell = -\frac{d\Phi}{dt} \tag{8}$$

Integrating this over the circle of the orbit, we get

$$2\pi rE = -\pi r^2\frac{dB}{dt} \tag{9}$$

The electric field causes a force on the electron:

$$F_E = qE = -\frac{qr}{2}\frac{dB}{dt} \tag{10}$$

Since $\mathbf{E}$ acts in the $+\phi$ direction, the force is parallel to the direction of motion. Over a time dt, this force will do work equal to $F_E\,ds$, where ds is the distance travelled in that time. Since the velocity changes from v to $v+dv$ and to first order, this change is linear, the average velocity over that time is $\frac{1}{2}(v+v+dv) = v+\frac{1}{2}dv$. The distance, again to first order, is then

$$ds = \left(v+\frac{1}{2}dv\right)dt = v\,dt \tag{11}$$

so the work done, which is equal to the change in kinetic energy, is

$$\begin{aligned} dT &= F_E\, ds && (12)\\ &= -\frac{qr}{2}\frac{dB}{dt}\,(v\, dt) && (13)\\ &= -\frac{qvr}{2}dB && (14) \end{aligned}$$

This is exactly the change in kinetic energy we worked out earlier, so the electric force provides the required work, provided that r doesn't change.