

MUTUAL INDUCTANCE: CALCULATION FROM THE INTEGRAL

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.52.

The general formula (known as the Neumann formula) for mutual inductance is

$$(0.1) \quad M = \frac{\mu_0}{4\pi} \int \int \frac{d\ell_1 \cdot d\ell_2}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

Although this formula isn't used much for direct calculation of M , we can revisit our initial mutual inductance problem, in which we have two parallel, circular loops sharing a common axis and separated by a distance z . The upper loop has radius a and the lower loop b . The integral is easiest if we use cylindrical coordinates. Both the line increments are in the ϕ direction with the upper increment having magnitude $d\ell_1 = a d\phi_a$ and the lower one $d\ell_2 = b d\phi_b$. The angle between the two increments is $\phi_a - \phi_b$ and both angles are integrated over the range $[0 \dots 2\pi]$. Therefore

$$(0.2) \quad d\ell_1 \cdot d\ell_2 = ab \cos(\phi_a - \phi_b) d\phi_a d\phi_b$$

For the denominator, we have

$$(0.3) \quad |\mathbf{r}_2 - \mathbf{r}_1| = \left[(a \cos \phi_a - b \cos \phi_b)^2 + (a \sin \phi_a - b \sin \phi_b)^2 + z^2 \right]^{1/2}$$

$$(0.4) \quad = \left[z^2 + a^2 + b^2 - 2ab \cos(\phi_a - \phi_b) \right]^{1/2}$$

If we define

$$(0.5) \quad \beta \equiv \frac{ab}{z^2 + a^2 + b^2}$$

we get

$$(0.6) \quad \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \sqrt{\frac{\beta}{ab}} \frac{1}{\sqrt{1 - 2\beta \cos(\phi_a - \phi_b)}}$$

We need to do the integral

$$(0.7) \quad M = \frac{\mu_0 \sqrt{ab\beta}}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(\phi_a - \phi_b) d\phi_a d\phi_b}{\sqrt{1 - 2\beta \cos(\phi_a - \phi_b)}}$$

This has no closed form solution, but we can expand the denominator in a Taylor series to get (defining $c \equiv \cos(\phi_a - \phi_b)$):

$$(0.8) \quad \frac{1}{\sqrt{1 - 2\beta \cos(\phi_a - \phi_b)}} = \frac{1}{\sqrt{1 - 2\beta c}}$$

$$(0.9) \quad = 1 + \beta c + \frac{3}{2}(\beta c)^2 + \frac{5}{2}(\beta c)^3 + \dots$$

The integral is then

$$(0.10) \quad M = \frac{\mu_0 \sqrt{ab\beta}}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \left[c + \beta c^2 + \frac{3}{2}\beta^2 c^3 + \frac{5}{2}\beta^3 c^4 + \dots \right] d\phi_a d\phi_b$$

$$(0.11) \quad = \frac{\mu_0 \sqrt{ab\beta}}{4\pi} \left[0 + 2\beta \pi^2 + 0 + \frac{15}{4}\beta^3 \pi^2 + \dots \right]$$

$$(0.12) \quad = \frac{\mu_0 \pi \beta \sqrt{ab\beta}}{2} \left[1 + \frac{15}{8}\beta^2 + \dots \right]$$

If the upper loop is much smaller than the lower one and the distance between them ($a \ll b$ and $a \ll z$), then to first order in a , we get $\beta \approx ab/(b^2 + z^2)$, $\beta^2 \approx 0$ and

$$(0.13) \quad M \approx \frac{\mu_0}{2} \frac{\pi a^2 b^2}{(b^2 + z^2)^{3/2}}$$

which agrees with the earlier result.