MUTUAL INDUCTANCE: CALCULATION FROM THE INTEGRAL

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.52.

The general formula (known as the Neumann formula) for mutual inductance is

$$M=\frac{\mu_{0}}{4\pi}\int\int\frac{d\ell_{1}\cdot d\ell_{2}}{|\mathbf{r}_{2}-\mathbf{r}_{1}|} \tag{1}$$

Although this formula isn't used much for direct calculation of M, we can revisit our initial mutual inductance problem, in which we have two parallel, circular loops sharing a common axis and separated by a distance z. The upper loop has radius a and the lower loop b. The integral is easiest if we use cylindrical coordinates. Both the line increments are in the ϕ direction with the upper increment having magnitude $d\ell_1 = a\, d\phi_a$ and the lower one $d\ell_2 = b\, d\phi_b$. The angle between the two increments is $\phi_a - \phi_b$ and both angles are integrated over the range $[0\ldots 2\pi]$. Therefore

$$d\ell_{1}\cdot d\ell_{2}=ab\cos\left(\phi_{a}-\phi_{b}\right)d\phi_{a}d\phi_{b} \tag{2}$$

For the denominator, we have

$$\begin{aligned}
|\mathbf{r}_{2}-\mathbf{r}_{1}| &= \left[\left(a\cos\phi_{a}-b\cos\phi_{b}\right)^{2}+\left(a\sin\phi_{a}-b\sin\phi_{b}\right)^{2}+z^{2}\right]^{1/2} && (3)\\
&= \left[z^{2}+a^{2}+b^{2}-2ab\cos\left(\phi_{a}-\phi_{b}\right)\right]^{1/2} && (4)
\end{aligned}$$

If we define

$$\beta\equiv\frac{ab}{z^{2}+a^{2}+b^{2}} \tag{5}$$

we get

$$\frac{1}{|\mathbf{r}_{2}-\mathbf{r}_{1}|}=\sqrt{\frac{\beta}{ab}}\frac{1}{\sqrt{1-2\beta\cos\left(\phi_{a}-\phi_{b}\right)}} \tag{6}$$

We need to do the integral

$$M=\frac{\mu_0\sqrt{ab\beta}}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(\phi_a-\phi_b)\,d\phi_a d\phi_b}{\sqrt{1-2\beta\cos(\phi_a-\phi_b)}} \tag{7}$$

This has no closed form solution, but we can expand the denominator in a Taylor series to get (defining $c\equiv\cos(\phi_a-\phi_b)$):

$$\frac{1}{\sqrt{1-2\beta\cos(\phi_a-\phi_b)}}=\frac{1}{\sqrt{1-2\beta c}} \tag{8}$$

$$=1+\beta c+\frac{3}{2}(\beta c)^2+\frac{5}{2}(\beta c)^3+\ldots \tag{9}$$

The integral is then

$$M=\frac{\mu_0\sqrt{ab\beta}}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\left[c+\beta c^2+\frac{3}{2}\beta^2c^3+\frac{5}{2}\beta^3c^4+\ldots\right]d\phi_a d\phi_b \tag{10}$$

$$=\frac{\mu_0\sqrt{ab\beta}}{4\pi}\left[0+2\beta\pi^2+0+\frac{15}{4}\beta^3\pi^2+\ldots\right] \tag{11}$$

$$=\frac{\mu_0\pi\beta\sqrt{ab\beta}}{2}\left[1+\frac{15}{8}\beta^2+\ldots\right] \tag{12}$$

If the upper loop is much smaller than the lower one and the distance between them ($a\ll b$ and $a\ll z$), then to first order in a, we get $\beta\approx ab/\left(b^2+z^2\right)$, $\beta^2\approx 0$ and

$$M\approx\frac{\mu_0}{2}\frac{\pi a^2b^2}{\left(b^2+z^2\right)^{3/2}} \tag{13}$$

which agrees with the earlier result.