TRANSFORMERS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 7.53, 7.54.

An electrical transformer (as opposed to Optimus Prime) is a device that uses mutual induction to transform one voltage and current to another with the same average power. To see how this works in a simplified model, suppose we have two wires that are wound around the same cylinder, with wire 1 having N_1 turns and wire 2 N_2 turns. If wire 1 is driven by an alternating voltage, then (assuming no resistance in wire 1), the back emf must balance the driving voltage, and is determined by

$$\mathscr{E}_1 = -\frac{d\Phi_1}{dt} \tag{0.1}$$

where Φ_1 is the total magnetic flux through wire 1. If the flux per turn is Φ then $\Phi_1 = N_1\Phi$ and

$$\mathscr{E}_1 = -N_1\frac{d\Phi}{dt} \tag{0.2}$$

If the windings of the two wires are arranged so that all loops of both wires contain the same flux, then the induced emf in wire 2 is

$$\mathscr{E}_2 = -\frac{d\Phi_2}{dt} = -N_2\frac{d\Phi}{dt} \tag{0.3}$$

so

$$\frac{\mathscr{E}_2}{\mathscr{E}_1} = \frac{N_2}{N_1} \tag{0.4}$$

That is, the voltage is stepped up or down, depending on the ratio of the number of turns in each coil. In order to conserve energy, the power $P = \mathscr{E}I$ must remain constant, so an increase in voltage means a decrease in current, and vice versa. To see this, suppose the driving voltage in wire 1 is

$$V_{in} = V_1 \cos\omega t \tag{0.5}$$

and that the self inductances of the two coils are L_1 and L_2, and their mutual inductance is M. Then the back emf in wire 1 must match V_{in}, and it results from the change in the current I_1 in wire 1 (via the self inductance L_1) *and* from the change in I_2 (via the mutual inductance M), so we get

$$-\mathscr{E}_1 = V_{in} = V_1 \cos \omega t = L_1 \dot{I}_1 + M \dot{I}_2 \tag{0.6}$$

If wire 2 is connected to a resistor R, then the voltage drop across the resistor must equal the back emf in wire 2 (as there is no driving voltage), so

$$-\mathscr{E}_2 = -V_{out} = -I_2 R = L_2 \dot{I}_2 + M \dot{I}_1 \tag{0.7}$$

To solve these two equations, we need a relation between M, L_1 and L_2. The self inductance per unit length of a solenoid of radius r and n turns per unit length is

$$L = \pi r^2 \mu_0 n^2 \tag{0.8}$$

so for our two intermeshed solenoids (assumed to be the same length ℓ):

$$L_{1,2} = \frac{\pi r^2}{\ell} \mu_0 N_{1,2}^2 \tag{0.9}$$

Using the same argument, the mutual inductance between two solenoids is found as follows. The field due to solenoid 1 is $B_1 = \mu_0 \frac{N_1}{\ell} I_1$. This produces a flux per turn through solenoid 2 of $\Phi_t = \pi r^2 \mu_0 \frac{N_1}{\ell} I_1$, for a total flux through 2 of $\Phi = \pi r^2 \mu_0 \frac{N_1}{\ell} N_2 I_1$. The mutual inductance is therefore

$$M = \pi r^2 \mu_0 \frac{N_1}{\ell} N_2 \tag{0.10}$$

Comparing the two, we see that

$$M^2 = L_1 L_2 \tag{0.11}$$

Substituting this into 0.6 and 0.7 above, we get

$$V_1 \cos \omega t = L_1 \dot{I}_1 + \sqrt{L_1 L_2} \dot{I}_2 \tag{0.12}$$

$$-I_2 R = L_2 \dot{I}_2 + \sqrt{L_1 L_2} \dot{I}_1 \tag{0.13}$$

Mutliplying the first equation by $\sqrt{L_2/L_1}$ and subtracting the second, we find

$$\sqrt{\frac{L_2}{L_1}}V_1\cos\omega t = -I_2R \tag{0.14}$$

$$I_2(t) = -\sqrt{\frac{L_2}{L_1}}\frac{V_1}{R}\cos\omega t \tag{0.15}$$

The current in wire 1 is then

$$\dot{I}_1 = \frac{V_1}{L_1}\cos\omega t - \sqrt{\frac{L_2}{L_1}}\dot{I}_2 \tag{0.16}$$

$$I_1(t) = \frac{V_1}{\omega L_1}\sin\omega t + \frac{L_2}{L_1}\frac{V_1}{R}\cos\omega t \tag{0.17}$$

assuming the constant of integration in the last line is zero (which means that I_1 has no DC component).

The voltage in wire 2 is thus

$$V_{out} = I_2R = -\sqrt{\frac{L_2}{L_1}}V_1\cos\omega t \tag{0.18}$$

The ratio of the two voltages is

$$\frac{V_{out}}{V_{in}} = -\sqrt{\frac{L_2}{L_1}} = -\frac{N_2}{N_1} \tag{0.19}$$

using 0.9.

Finally, we can work out the power in each wire.

$$P_1 = I_1V_{in} \tag{0.20}$$

$$= \frac{V_1^2}{\omega L_1}\sin\omega t\cos\omega t + \frac{L_2}{L_1}\frac{V_1^2}{R}\cos^2\omega t \tag{0.21}$$

$$P_2 = I_2V_{out} \tag{0.22}$$

$$= \frac{L_2}{L_1}\frac{V_1^2}{R}\cos^2\omega t \tag{0.23}$$

If we integrate these two powers over a full cycle (for $t = 0\ldots 2\pi/\omega$) the first term in P_1 integrates to zero and the last term in P_1 equals P_2 so the average power in each wire is the same.

$$\langle P_1 \rangle = \langle P_2 \rangle = \frac{L_2}{L_1} \frac{V_1^2}{R} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos^2 \omega t \, dt = \frac{L_2}{2L_1} \frac{V_1^2}{R} \tag{0.24}$$