

## TRANSFORMERS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 7.53, 7.54.

An electrical transformer (as opposed to Optimus Prime) is a device that uses mutual induction to transform one voltage and current to another with the same average power. To see how this works in a simplified model, suppose we have two wires that are wound around the same cylinder, with wire 1 having  $N_1$  turns and wire 2  $N_2$  turns. If wire 1 is driven by an alternating voltage, then (assuming no resistance in wire 1), the back emf must balance the driving voltage, and is determined by

$$(1) \quad \mathcal{E}_1 = -\frac{d\Phi_1}{dt}$$

where  $\Phi_1$  is the total magnetic flux through wire 1. If the flux per turn is  $\Phi$  then  $\Phi_1 = N_1\Phi$  and

$$(2) \quad \mathcal{E}_1 = -N_1\frac{d\Phi}{dt}$$

If the windings of the two wires are arranged so that all loops of both wires contain the same flux, then the induced emf in wire 2 is

$$(3) \quad \mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -N_2\frac{d\Phi}{dt}$$

so

$$(4) \quad \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}$$

That is, the voltage is stepped up or down, depending on the ratio of the number of turns in each coil. In order to conserve energy, the power  $P = \mathcal{E}I$  must remain constant, so an increase in voltage means a decrease in current, and vice versa. To see this, suppose the driving voltage in wire 1 is

$$(5) \quad V_{in} = V_1 \cos \omega t$$

and that the self inductances of the two coils are  $L_1$  and  $L_2$ , and their mutual inductance is  $M$ . Then the back emf in wire 1 must match  $V_{in}$ , and it results from the change in the current  $I_1$  in wire 1 (via the self inductance  $L_1$ ) and from the change in  $I_2$  (via the mutual inductance  $M$ ), so we get

$$(6) \quad -\mathcal{E}_1 = V_{in} = V_1 \cos \omega t = L_1 \dot{I}_1 + M \dot{I}_2$$

If wire 2 is connected to a resistor  $R$ , then the voltage drop across the resistor must equal the back emf in wire 2 (as there is no driving voltage), so

$$(7) \quad -\mathcal{E}_2 = -V_{out} = -I_2 R = L_2 \dot{I}_2 + M \dot{I}_1$$

To solve these two equations, we need a relation between  $M$ ,  $L_1$  and  $L_2$ . The self inductance per unit length of a solenoid of radius  $r$  and  $n$  turns per unit length is

$$(8) \quad L = \pi r^2 \mu_0 n^2$$

so for our two intermeshed solenoids (assumed to be the same length  $\ell$ ):

$$(9) \quad L_{1,2} = \frac{\pi r^2}{\ell} \mu_0 N_{1,2}^2$$

Using the same argument, the mutual inductance between two solenoids is found as follows. The field due to solenoid 1 is  $B_1 = \mu_0 \frac{N_1}{\ell} I_1$ . This produces a flux per turn through solenoid 2 of  $\Phi_t = \pi r^2 \mu_0 \frac{N_1}{\ell} I_1$ , for a total flux through 2 of  $\Phi = \pi r^2 \mu_0 \frac{N_1}{\ell} N_2 I_1$ . The mutual inductance is therefore

$$(10) \quad M = \pi r^2 \mu_0 \frac{N_1}{\ell} N_2$$

Comparing the two, we see that

$$(11) \quad M^2 = L_1 L_2$$

Substituting this into 6 and 7 above, we get

$$(12) \quad V_1 \cos \omega t = L_1 \dot{I}_1 + \sqrt{L_1 L_2} \dot{I}_2$$

$$(13) \quad -I_2 R = L_2 \dot{I}_2 + \sqrt{L_1 L_2} \dot{I}_1$$

Multiplying the first equation by  $\sqrt{L_2/L_1}$  and subtracting the second, we find

$$(14) \quad \sqrt{\frac{L_2}{L_1}} V_1 \cos \omega t = -I_2 R$$

$$(15) \quad I_2(t) = -\sqrt{\frac{L_2}{L_1}} \frac{V_1}{R} \cos \omega t$$

The current in wire 1 is then

$$(16) \quad \dot{I}_1 = \frac{V_1}{L_1} \cos \omega t - \sqrt{\frac{L_2}{L_1}} \dot{I}_2$$

$$(17) \quad I_1(t) = \frac{V_1}{\omega L_1} \sin \omega t + \frac{L_2 V_1}{L_1 R} \cos \omega t$$

assuming the constant of integration in the last line is zero (which means that  $I_1$  has no DC component).

The voltage in wire 2 is thus

$$(18) \quad V_{out} = I_2 R = -\sqrt{\frac{L_2}{L_1}} V_1 \cos \omega t$$

The ratio of the two voltages is

$$(19) \quad \frac{V_{out}}{V_{in}} = -\sqrt{\frac{L_2}{L_1}} = -\frac{N_2}{N_1}$$

using 9.

Finally, we can work out the power in each wire.

$$(20) \quad P_1 = I_1 V_{in}$$

$$(21) \quad = \frac{V_1^2}{\omega L_1} \sin \omega t \cos \omega t + \frac{L_2 V_1^2}{L_1 R} \cos^2 \omega t$$

$$(22) \quad P_2 = I_2 V_{out}$$

$$(23) \quad = \frac{L_2 V_1^2}{L_1 R} \cos^2 \omega t$$

If we integrate these two powers over a full cycle (for  $t = 0 \dots 2\pi/\omega$ ) the first term in  $P_1$  integrates to zero and the last term in  $P_1$  equals  $P_2$  so the average power in each wire is the same.

$$(24) \quad \langle P_1 \rangle = \langle P_2 \rangle = \frac{L_2 V_1^2}{L_1 R} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos^2 \omega t \, dt = \frac{L_2 V_1^2}{2L_1 R}$$