MAXWELL'S EQUATIONS WITH VARYING CHARGE BUT CONSTANT CURRENT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 7.55.

An interesting situation in electrodynamics is one where the current density $\mathbf{J}$ is constant in time, but the charge density ρ isn't. Clearly this means that charge is piling up somewhere, which could happen when charging a capacitor. However, the rate at which ρ changes is constrained in this situation, as we can show using Maxwell's equations. First, from Gauss's law:

$$\nabla\cdot\dot{\mathbf{E}}=\frac{1}{\varepsilon_0}\dot{\rho} \tag{1}$$

Then from Faraday's law

$$\nabla\cdot(\nabla\times\mathbf{B})=\mu_0\nabla\cdot\mathbf{J}+\mu_0\varepsilon_0\nabla\cdot\dot{\mathbf{E}} \tag{2}$$

Since the divergence of a curl is always zero, we get

$$\nabla\cdot\dot{\mathbf{E}} = -\frac{1}{\varepsilon_0}\nabla\cdot\mathbf{J} \tag{3}$$

$$\dot{\rho} = -\nabla\cdot\mathbf{J} \tag{4}$$

Since $\mathbf{J}$ is independent of time, we can integrate this directly to get

$$\rho(\mathbf{r},t)=-\nabla\cdot\mathbf{J}(\mathbf{r})\,t+\rho(\mathbf{r},0) \tag{5}$$

so the charge density is a linear function of the time.

In this situation, the Biot-Savart law still holds, as we can see by using Maxwell's equations. The law is

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{6}$$

First, we can check the extended version of Ampère's law:

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{7}$$

Taking the curl of the Biot-Savart law, we get

$$\nabla_r\times\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\nabla_r\times\int\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{8}$$

where the subscript r reminds us that the curl is with respect to the unprimed coordinates.

Using a product rule we get

$$\nabla_r\times\left[\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right]=-\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla\right)\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}+\mathbf{J}\left(\mathbf{r}'\right)\left(\nabla\cdot\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right) \tag{9}$$

The divergence in the second term on the RHS can be written in terms of the 3-d delta function. Again, remember that we're taking the derivative only with respect to $\mathbf{r}$ components, so the presence of the $\mathbf{r}'$ serves merely to shift the origin, so we have

$$\nabla\cdot\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}=4\pi\delta_3\left(\mathbf{r}-\mathbf{r}'\right) \tag{10}$$

We can work out the integral of the first term on the RHS using the same technique that was applied to the original, magnetostatic version of Ampère's law. If you refer back to that derivation, we transformed the x component of the integrand to the form

$$-\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla_{r'}\right)\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}=-\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}\nabla_{r'}\cdot\mathbf{J}\left(\mathbf{r}'\right)+\nabla_{r'}\cdot\left(\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{J}\left(\mathbf{r}'\right)\right) \tag{11}$$

where the derivatives are now with respect to the primed coordinates. The second term on the RHS can be converted to a surface integral using the divergence theorem and, if the currents are localized, works out to zero. In the magnetostatic case $\nabla_{r'}\cdot\mathbf{J}(\mathbf{r}')=0$, but here that isn't true, as we saw above. However, if we return to three dimensions are look again at the integral, we get

$$\begin{aligned}-\int\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\nabla_{r'}\cdot\mathbf{J}\left(\mathbf{r}'\right)d^3\mathbf{r}' &= \int\dot{\rho}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' && (12)\\ &= \frac{\partial}{\partial t}\int\rho\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' && (13)\end{aligned}$$

However, the integral is essentially the definition of the electric field:

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int\rho\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{14}$$

Combining this with 10, we get

$$\nabla\cdot(\nabla\times\mathbf{B})=\mu_0\nabla\cdot\mathbf{J}+\mu_0\varepsilon_0\nabla\cdot\dot{\mathbf{E}} \tag{15}$$

which is just Ampère's law again.

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