DISPLACEMENT CURRENT IN AN INFINITE WIRE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 7.56.

Terry, William K. (1982), Am. J. Phys. **50**, 742.

This is an illustration of the role of Maxwell's displacement current in Ampère's law

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\mathbf{J}_d \tag{1}$$

with the displacement current defined as

$$\mathbf{J}_d\equiv\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{2}$$

Consider the case of an infinite straight wire carrying a current I. This is a magnetostatic case, and as such we can calculate the magnetic field from Ampère's law with $\partial\mathbf{E}/\partial t=0$ and get the usual formula

$$\mathbf{B}=\frac{\mu_0 I}{2\pi r}\hat{\phi} \tag{3}$$

In this case the displacement current is zero. Suppose, however, we introduce a small gap of width ϵ in the current. (This is an idealized situation in which we picture a moving line of charge where the charge density is λ everywhere except in the gap, where there is no charge.) We can simulate this situation by considering a tiny segment of wire of length ϵ carrying the opposite charge density $-\lambda$ travelling at a speed v, where $I=\lambda v$. Superimposing this wire segment onto the infinite wire gives a small gap travelling along at speed v. Because of the small gap in the wire, $\partial\mathbf{E}/\partial t$ is no longer zero as this segment moves along. We'll place the wire along the z axis and calculate the field in the xy plane. From the integral form of Ampère's law we have, for an integration path around a circle of radius R in the xy plane:

$$\frac{2\pi}{\mu_0}RB=\int\mathbf{J}\cdot d\mathbf{a}+\frac{\partial}{\partial t}\int\mathbf{E}\cdot d\mathbf{a} \tag{4}$$

Consider a time when the gap straddles the origin. The net current across the surface is then $\mathbf{J}=0$, so any contribution to the magnetic field must come from the changing electric field. The infinite wire produces a constant, radial electric field so contributes nothing to $\partial\mathbf{E}/\partial t$, so we can look at the field due to the short segment of wire.

For a segment of wire of length dz at position z, the z component of field (all we're interested in, since it's $\mathbf{E}\cdot d\mathbf{a}$ we have to calculate) at a point on a circle of radius s in the xy plane is

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{-\lambda\,dz}{(s^2+z^2)}\frac{-z}{\sqrt{s^2+z^2}} \tag{5}$$

We've used $-z$ in the numerator to get the direction of the field right. If dz is in the region $z<0$ then E_z is negative (points towards $-z$) since the charge is negative. If we take $d\mathbf{a}$ to point in the $+z$ direction, we want $\mathbf{E}\cdot d\mathbf{a}<0$ in this case. Similarly, if $z>0$, E_z should be positive, since the field points towards $+z$. Therefore

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda z\,dz}{\left(s^2+z^2\right)^{3/2}} \tag{6}$$

The total field at a point in the xy plane is thus (assuming that the leading edge of the segment crosses $z=0$ at $t=0$):

$$\begin{aligned} E_z &= \frac{\lambda}{4\pi\epsilon_0}\int_{vt-\epsilon}^{vt}\frac{z\,dz}{\left(s^2+z^2\right)^{3/2}} && (7)\\ &= \frac{\lambda}{4\pi\epsilon_0}\left[\left(s^2+(vt-\epsilon)^2\right)^{-1/2}-\left(s^2+(vt)^2\right)^{-1/2}\right] && (8)\end{aligned}$$

The total flux through the circle from the segment is then

$$\begin{aligned} \Phi_E &= \int \mathbf{E}\cdot d\mathbf{a} && (9)\\ &= \frac{2\pi\lambda}{4\pi\epsilon_0}\int_0^R\left[\left(s^2+(vt-\epsilon)^2\right)^{-1/2}-\left(s^2+(vt)^2\right)^{-1/2}\right]s\,ds && (10)\\ &= \frac{\lambda}{2\epsilon_0}\left[\sqrt{s^2+(vt-\epsilon)^2}-\sqrt{s^2+(vt)^2}\right]\Bigg|_0^R && (11)\end{aligned}$$

At this point we're faced with a bit of a conundrum, for when evaluating the $s=0$ limit, we need to choose whether to take the positive or negative square root. If the segment straddles the origin and we take the positive root in both cases, then we get

$$\Phi_E = \frac{\lambda}{2\epsilon_0}\left[\sqrt{R^2+(vt-\epsilon)^2}-\sqrt{R^2+(vt)^2}-(\epsilon-vt)+vt\right] \tag{12}$$

$$= \frac{\lambda}{2\epsilon_0}\left[\sqrt{R^2+(vt-\epsilon)^2}-\sqrt{R^2+(vt)^2}-\epsilon+2vt\right] \tag{13}$$

This result looks a bit dubious since the flux doesn't go to zero when $\epsilon \to 0$, which you would intuitively think it must. The catch appears to be that if we're requiring the segment to straddle the origin, then the integrand has a singularity when $s = vt = 0$. Indeed, in the original paper by Terry, he doesn't derive the result this way; rather he begins with a point charge moving along the z axis with the result that the flux contains a step function when the charge passes through the circle. This in turn gives the displacement current a delta function at the origin.

In any event, the displacement current here is

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt} \tag{14}$$

$$= \lambda v + \frac{\lambda v^2}{2}\left[\frac{t}{\sqrt{R^2+(vt-\epsilon)^2}}-\frac{t}{\sqrt{R^2+(vt)^2}}\right] \tag{15}$$

As $\epsilon \to 0$, $I_d \to \lambda v = I$, so in theory, the displacement current provides the magnetic field in the gap.

I have to confess that I don't find this derivation particularly convincing, due to the fudging of the limits of integration and the sweeping under the carpet of singularities. Terry's original treatment is much clearer.