

DISPLACEMENT CURRENT IN AN INFINITE WIRE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 7.56.

Terry, William K. (1982), Am. J. Phys. **50**, 742.

This is an illustration of the role of Maxwell's displacement current in Ampère's law

$$(1) \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \mathbf{J}_d$$

with the displacement current defined as

$$(2) \quad \mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Consider the case of an infinite straight wire carrying a current I . This is a magnetostatic case, and as such we can calculate the magnetic field from Ampère's law with $\partial \mathbf{E} / \partial t = 0$ and get the usual formula

$$(3) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

In this case the displacement current is zero. Suppose, however, we introduce a small gap of width ϵ in the current. (This is an idealized situation in which we picture a moving line of charge where the charge density is λ everywhere except in the gap, where there is no charge.) We can simulate this situation by considering a tiny segment of wire of length ϵ carrying the opposite charge density $-\lambda$ travelling at a speed v , where $I = \lambda v$. Superimposing this wire segment onto the infinite wire gives a small gap travelling along at speed v . Because of the small gap in the wire, $\partial \mathbf{E} / \partial t$ is no longer zero as this segment moves along. We'll place the wire along the z axis and calculate the field in the xy plane. From the integral form of Ampère's law we have, for an integration path around a circle of radius R in the xy plane:

$$(4) \quad \frac{2\pi}{\mu_0} RB = \int \mathbf{J} \cdot d\mathbf{a} + \frac{\partial}{\partial t} \int \mathbf{E} \cdot d\mathbf{a}$$

Consider a time when the gap straddles the origin. The net current across the surface is then $\mathbf{J} = 0$, so any contribution to the magnetic field must come from the changing electric field. The infinite wire produces a constant, radial electric field so contributes nothing to $\partial\mathbf{E}/\partial t$, so we can look at the field due to the short segment of wire.

For a segment of wire of length dz at position z , the z component of field (all we're interested in, since it's $\mathbf{E} \cdot d\mathbf{a}$ we have to calculate) at a point on a circle of radius s in the xy plane is

$$(5) \quad dE_z = \frac{1}{4\pi\epsilon_0} \frac{-\lambda dz}{(s^2 + z^2)} \frac{-z}{\sqrt{s^2 + z^2}}$$

We've used $-z$ in the numerator to get the direction of the field right. If dz is in the region $z < 0$ then E_z is negative (points towards $-z$) since the charge is negative. If we take $d\mathbf{a}$ to point in the $+z$ direction, we want $\mathbf{E} \cdot d\mathbf{a} < 0$ in this case. Similarly, if $z > 0$, E_z should be positive, since the field points towards $+z$. Therefore

$$(6) \quad dE_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda z dz}{(s^2 + z^2)^{3/2}}$$

The total field at a point in the xy plane is thus (assuming that the leading edge of the segment crosses $z = 0$ at $t = 0$):

$$(7) \quad E_z = \frac{\lambda}{4\pi\epsilon_0} \int_{vt-\epsilon}^{vt} \frac{z dz}{(s^2 + z^2)^{3/2}}$$

$$(8) \quad = \frac{\lambda}{4\pi\epsilon_0} \left[\left(s^2 + (vt - \epsilon)^2 \right)^{-1/2} - \left(s^2 + (vt)^2 \right)^{-1/2} \right]$$

The total flux through the circle from the segment is then

$$(9) \quad \Phi_E = \int \mathbf{E} \cdot d\mathbf{a}$$

$$(10) \quad = \frac{2\pi\lambda}{4\pi\epsilon_0} \int_0^R \left[\left(s^2 + (vt - \epsilon)^2 \right)^{-1/2} - \left(s^2 + (vt)^2 \right)^{-1/2} \right] s ds$$

$$(11) \quad = \frac{\lambda}{2\epsilon_0} \left[\sqrt{s^2 + (vt - \epsilon)^2} - \sqrt{s^2 + (vt)^2} \right] \Big|_0^R$$

At this point we're faced with a bit of a conundrum, for when evaluating the $s = 0$ limit, we need to choose whether to take the positive or negative square root. If the segment straddles the origin and we take the positive root in both cases, then we get

$$(12) \quad \Phi_E = \frac{\lambda}{2\epsilon_0} \left[\sqrt{R^2 + (vt - \epsilon)^2} - \sqrt{R^2 + (vt)^2} - (\epsilon - vt) + vt \right]$$

$$(13) \quad = \frac{\lambda}{2\epsilon_0} \left[\sqrt{R^2 + (vt - \epsilon)^2} - \sqrt{R^2 + (vt)^2} - \epsilon + 2vt \right]$$

This result looks a bit dubious since the flux doesn't go to zero when $\epsilon \rightarrow 0$, which you would intuitively think it must. The catch appears to be that if we're requiring the segment to straddle the origin, then the integrand has a singularity when $s = vt = 0$. Indeed, in the original paper by Terry, he doesn't derive the result this way; rather he begins with a point charge moving along the z axis with the result that the flux contains a step function when the charge passes through the circle. This in turn gives the displacement current a delta function at the origin.

In any event, the displacement current here is

$$(14) \quad I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

$$(15) \quad = \lambda v + \frac{\lambda v^2}{2} \left[\frac{t}{\sqrt{R^2 + (vt - \epsilon)^2}} - \frac{t}{\sqrt{R^2 + (vt)^2}} \right]$$

As $\epsilon \rightarrow 0$, $I_d \rightarrow \lambda v = I$, so in theory, the displacement current provides the magnetic field in the gap.

I have to confess that I don't find this derivation particularly convincing, due to the fudging of the limits of integration and the sweeping under the carpet of singularities. Terry's original treatment is much clearer.