ELECTRIC FIELD OUTSIDE AN INFINITE WIRE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 7.57.

The electric field inside a resistor with a constant cross-section is given in Griffiths's Example 7.1 as

$$\mathbf{E}=\frac{I\rho}{A}\hat{\mathbf{z}} \tag{1}$$

where in this case ρ is the resistivity (*not* the charge density!), I is the current and A is the cross-sectional area of the resistor. If we consider the simple case where the cross section is circular and make the resistor very long (so we can call it a wire with radius a), then

$$\mathbf{E}=\frac{I\rho}{\pi a^2}\hat{\mathbf{z}} \tag{2}$$

Surprisingly, the problem of finding the field outside the wire is not completely solved. If we assume that the current returns through a superconducting coaxial cylinder of radius b around the wire, and that the magnetic field is constant in time, so that $\nabla\times\mathbf{E}=0$ and thus $\mathbf{E}=-\nabla V$, then we can apply Laplace's equation in cylindrical coordinates to find the potential. In order to solve it, we need boundary conditions. Since the coaxial cylinder is a perfect conductor, $V=0$ there. On the surface of the inner wire, we have

$$V(a,z)=-\frac{I\rho z}{\pi a^2} \tag{3}$$

The problem is, what is V in between the two cylinders? The solution we derived earlier was for the case where V was independent of z:

(4)

$$V(r,\phi)=A\ln r+B+\sum_{n=1}^{\infty}r^n\left(A_n\sin n\phi+B_n\cos n\phi\right)+\sum_{n=-\infty}^{-1}r^n\left(C_n\sin n\phi+D_n\cos n\phi\right)$$

However, in the special case where $V(r,z) = zf(r)$, this solution also works for finding f (since using separation of variables on all three cylindrical variables separates the z from the rest of V). Since there is no ϕ dependence, both sums disappear and we are left with

$$V(r=b) = 0 = A\ln b + B \tag{5}$$

$$B = -A\ln b \tag{6}$$

On the wire:

$$V(a) = -\frac{I\rho z}{\pi a^2} \tag{7}$$

$$= A\ln a + B \tag{8}$$

$$= A\ln\frac{a}{b} \tag{9}$$

$$A = -\frac{I\rho z}{\pi a^2 \ln(a/b)} \tag{10}$$

So

$$V(r,z) = -\frac{I\rho z \ln(r/b)}{\pi a^2 \ln(a/b)} \tag{11}$$

The field is then

$$\mathbf{E} = -\nabla V \tag{12}$$

$$= \frac{I\rho}{\pi a^2 \ln(a/b)}\left[\frac{z}{r}\hat{\mathbf{r}} + \ln\frac{r}{b}\hat{\mathbf{z}}\right] \tag{13}$$

The surface charge density on the wire is found from the difference in radial components of the field. Inside the wire there is no radial component, so

$$\sigma = \varepsilon_0 \Delta E_r = \frac{I\rho\varepsilon_0 z}{\pi a^3 \ln(a/b)} \tag{14}$$

As Griffiths remarks, the results for the field and surface charge are 'peculiar' (I would say 'wrong') since they depend on z, which shouldn't be the case for an infinite wire with no variation along its length. It is surely impossible since the results depend on where we set the origin, which is completely arbitrary.