ALFVEN'S THEOREM

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 7.59.

Starting from Ohm's law in the form

$$\mathbf{J}=\sigma\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) \tag{1}$$

we can derive Alfven's theorem, which states that in a perfectly conducting fluid, the magnetic field lines get frozen, which means that if we take some closed loop within the fluid, the magnetic flux through that loop remains constant as the loop gets carried along by the fluid.

Since the fluid is a perfect conductor, the conductance $\sigma=\infty$ so if $\mathbf{J}$ is finite, we must have

$$\mathbf{E}=-\mathbf{v}\times\mathbf{B} \tag{2}$$

Applying Faraday's law we get

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\left(\mathbf{v}\times\mathbf{B}\right) \tag{3}$$

Now consider a loop $\mathscr{P}$ enclosing a surface $\mathscr{S}$ at time t. Over a time interval dt, $\mathscr{S}$ moves a distance $\mathbf{v}\,dt$ with the fluid, and ends up as a surface $\mathscr{S}'$ surrounded by loop $\mathscr{P}'$. The surface between $\mathscr{S}$ and $\mathscr{S}'$ forms a ribbon $\mathscr{R}$. The change in flux through the loop is then

$$d\Phi=\int_{\mathscr{S}'}\mathbf{B}\left(t+dt\right)\cdot d\mathbf{a}-\int_{\mathscr{S}}\mathbf{B}\left(t\right)\cdot d\mathbf{a} \tag{4}$$

The three surfaces $-\mathscr{S}$ (the minus means we are reversing the surface normal so it points outward from the combined volume), $\mathscr{S}'$ and $\mathscr{R}$ taken together form a closed surface, and if we integrate $\mathbf{B}\left(t+dt\right)$ over this closed surface, we can use the divergence theorem to say:

$$\int_{-\mathscr{S}+\mathscr{S}'+\mathscr{R}}\mathbf{B}\cdot d\mathbf{a}=\int_{V}\nabla\cdot\mathbf{B}d^{3}\mathbf{r}=0 \tag{5}$$

since $\nabla\cdot\mathbf{B}=0$ always. Therefore

$$\int_{\mathscr{S}'}\mathbf{B}(t+dt)\cdot d\mathbf{a}+\int_{\mathscr{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a}=\int_{\mathscr{S}}\mathbf{B}(t+dt)\cdot d\mathbf{a} \tag{6}$$

Plugging this back into 4 we get

$$d\Phi=\int_{\mathscr{S}}\mathbf{B}(t+dt)\cdot d\mathbf{a}-\int_{\mathscr{S}}\mathbf{B}(t)\cdot d\mathbf{a}-\int_{\mathscr{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a} \tag{7}$$

$$=dt\int_{\mathscr{S}}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a}-\int_{\mathscr{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a} \tag{8}$$

Keep in mind that the ribbon $\mathscr{R}$ is a differential area, so the differentials in this equation still balance out.

The differential area is a parallelogram bounded by a line element $d\ell$ around the loop $\mathscr{P}$ and the vector $\mathbf{v}dt$, so

$$d\mathbf{a}=d\ell\times\mathbf{v}dt \tag{9}$$

The integral over $d\mathbf{a}$ is thus equivalent to adding up these area elements as we move around the loop $\mathscr{P}$. That is

$$\int_{\mathscr{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a}=\oint_{\mathscr{P}}\mathbf{B}(t+dt)\cdot(d\ell\times\mathbf{v}dt) \tag{10}$$

To first order in dt, this integrand is

$$\mathbf{B}(t+dt)\cdot(d\ell\times\mathbf{v}dt)=\mathbf{B}(t)\cdot(d\ell\times\mathbf{v}dt)+\mathscr{O}\left(dt^2\right) \tag{11}$$

so we can write it as

$$\int_{\mathscr{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a}=dt\oint_{\mathscr{P}}\mathbf{B}(t)\cdot(d\ell\times\mathbf{v}) \tag{12}$$

Using the vector triple product rule (we can cyclically permute the three vectors), we get

$$dt\oint_{\mathscr{P}}\mathbf{B}(t)\cdot(d\ell\times\mathbf{v})=dt\oint_{\mathscr{P}}d\ell\cdot(\mathbf{v}\times\mathbf{B}(t)) \tag{13}$$

Since we're now integrating $\mathbf{v}\times\mathbf{B}$ around the boundary of surface $\mathscr{S}$, we can use Stokes's theorem:

$$dt\oint_{\mathscr{P}} d\ell\cdot\left(\mathbf{v}\times\mathbf{B}\left(t\right)\right)=dt\int_{\mathscr{S}}\nabla\times\left(\mathbf{v}\times\mathbf{B}\left(t\right)\right)\cdot d\mathbf{a} \tag{14}$$

We therefore get finally:

$$d\Phi=dt\int_{\mathscr{S}}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a}-dt\int_{\mathscr{S}}\nabla\times\left(\mathbf{v}\times\mathbf{B}\left(t\right)\right)\cdot d\mathbf{a} \tag{15}$$

$$\frac{d\Phi}{dt}=\int_{\mathscr{S}}\left[\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a}-\nabla\times\left(\mathbf{v}\times\mathbf{B}\left(t\right)\right)\right]\cdot d\mathbf{a} \tag{16}$$

Using 3 we get finally

$$\frac{d\Phi}{dt}=0 \tag{17}$$

so the flux through the loop doesn't change.