DUALITY TRANSFORMATION FOR MAGNETIC AND ELECTRIC CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 7, Post 60.

Continuing our excursion in the fantasy world where magnetic monopoles exist, we can explore the *duality transformations*, which describe a rotation in an E-B space, as follows:

$$\mathbf{E}' = \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha \tag{1}$$

$$c\mathbf{B}' = c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha \tag{2}$$

$$cq_e' = cq_e\cos\alpha + q_m\sin\alpha \tag{3}$$

$$q_m' = q_m\cos\alpha - cq_e\sin\alpha \tag{4}$$

where α is an arbitrary angle and $c = 1/\sqrt{\mu_0\epsilon_0}$. Currents and charge densities transform the same way as the corresponding point charges.

It's a straightforward, though tedious, exercise to verify that Maxwell's equations (including magnetic charge) are invariant under these transformations. The two equations that are different are:

$$\nabla\cdot\mathbf{B} = \mu_0\rho_m \tag{5}$$

$$\nabla\times\mathbf{E} = -\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t} \tag{6}$$

A few sample calculations should show how it goes. We'll do one involving a divergence:

$$c\nabla\cdot\mathbf{B}' = c\nabla\cdot\mathbf{B}\cos\alpha - \nabla\cdot\mathbf{E}\sin\alpha \tag{7}$$

$$= c\mu_0\rho_m\cos\alpha - \frac{\rho_e}{\epsilon_0}\sin\alpha \tag{8}$$

$$= c\mu_0\left(\rho_m\cos\alpha - c\rho_e\sin\alpha\right) \tag{9}$$

$$= c\mu_0\rho_m' \tag{10}$$

And another involving a curl:

$$\nabla\times\mathbf{E}'=\nabla\times\mathbf{E}\cos\alpha+c\nabla\times\mathbf{B}\sin\alpha \tag{11}$$
$$=\left(-\mu_0\mathbf{J}_m-\frac{\partial\mathbf{B}}{\partial t}\right)\cos\alpha+c\left(\mu_0\mathbf{J}_e+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\sin\alpha \tag{12}$$
$$=-\mu_0\left(\mathbf{J}_m\cos\alpha-c\mathbf{J}_e\sin\alpha\right)-\frac{\partial\mathbf{B}}{\partial t}\cos\alpha+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\sin\alpha \tag{13}$$
$$=-\mu_0\mathbf{J}_m'-\frac{\partial\mathbf{B}'}{\partial t} \tag{14}$$

The other two equations transform similarly.

If we take the force law for electric and magnetic charges to be

$$\mathbf{F}=q_e\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)+q_m\left(\mathbf{B}-\frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) \tag{15}$$

then the transformed law is

$$\mathbf{F}'=q_e'\left(\mathbf{E}'+\mathbf{v}\times\mathbf{B}'\right)+q_m'\left(\mathbf{B}'-\frac{1}{c^2}\mathbf{v}\times\mathbf{E}'\right) \tag{16}$$
$$=\left(q_e\cos\alpha+\frac{q_m}{c}\sin\alpha\right)\times \tag{17}$$
$$\left[\mathbf{E}\cos\alpha+c\mathbf{B}\sin\alpha+\mathbf{v}\times\left(\mathbf{B}\cos\alpha-\frac{1}{c}\mathbf{E}\sin\alpha\right)\right]+ \tag{18}$$
$$\left(q_m\cos\alpha-cq_e\sin\alpha\right)\times \tag{19}$$
$$\left[\mathbf{B}\cos\alpha-\frac{1}{c}\mathbf{E}\sin\alpha-\frac{1}{c^2}\mathbf{v}\times\left(\mathbf{E}\cos\alpha+c\mathbf{B}\sin\alpha\right)\right] \tag{20}$$
$$=q_e\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)+q_m\left(\mathbf{B}-\frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) \tag{21}$$
$$=\mathbf{F} \tag{22}$$

To get the penultimate line it is just a matter of multiplying out the first expression and using $\cos^2\alpha+\sin^2\alpha=1$, and cancelling off terms.

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