DUALITY TRANSFORMATION FOR MAGNETIC AND ELECTRIC CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 7, Post 60.

Continuing our excursion in the fantasy world where magnetic monopoles exist, we can explore the *duality transformations*, which describe a rotation in an E-B space, as follows:

$$\begin{aligned}
\mathbf{E}' &= \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha & (1)\\
c\mathbf{B}' &= c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha & (2)\\
cq_e' &= cq_e\cos\alpha + q_m\sin\alpha & (3)\\
q_m' &= q_m\cos\alpha - cq_e\sin\alpha & (4)
\end{aligned}$$

where α is an arbitrary angle and $c = 1/\sqrt{\mu_0\epsilon_0}$. Currents and charge densities transform the same way as the corresponding point charges.

It's a straightforward, though tedious, exercise to verify that Maxwell's equations (including magnetic charge) are invariant under these transformations. The two equations that are different are:

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \mu_0\rho_m & (5)\\
\nabla\times\mathbf{E} &= -\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t} & (6)
\end{aligned}$$

A few sample calculations should show how it goes. We'll do one involving a divergence:

$$\begin{aligned}
c\nabla\cdot\mathbf{B}' &= c\nabla\cdot\mathbf{B}\cos\alpha - \nabla\cdot\mathbf{E}\sin\alpha & (7)\\
&= c\mu_0\rho_m\cos\alpha - \frac{\rho_e}{\epsilon_0}\sin\alpha & (8)\\
&= c\mu_0\left(\rho_m\cos\alpha - c\rho_e\sin\alpha\right) & (9)\\
&= c\mu_0\rho_m' & (10)
\end{aligned}$$

And another involving a curl:

$$\nabla\times\mathbf{E}' = \nabla\times\mathbf{E}\cos\alpha + c\nabla\times\mathbf{B}\sin\alpha \tag{11}$$
$$= \left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\cos\alpha + c\left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\sin\alpha \tag{12}$$
$$= -\mu_0\left(\mathbf{J}_m\cos\alpha - c\mathbf{J}_e\sin\alpha\right) - \frac{\partial\mathbf{B}}{\partial t}\cos\alpha + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\sin\alpha \tag{13}$$
$$= -\mu_0\mathbf{J}'_m - \frac{\partial\mathbf{B}'}{\partial t} \tag{14}$$

The other two equations transform similarly.

If we take the force law for electric and magnetic charges to be

$$\mathbf{F} = q_e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) \tag{15}$$

then the transformed law is

$$\mathbf{F}' = q'_e\left(\mathbf{E}' + \mathbf{v}\times\mathbf{B}'\right) + q'_m\left(\mathbf{B}' - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}'\right) \tag{16}$$
$$= \left(q_e\cos\alpha + \frac{q_m}{c}\sin\alpha\right)\times \tag{17}$$
$$\left[\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha + \mathbf{v}\times\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right)\right] + \tag{18}$$
$$\left(q_m\cos\alpha - cq_e\sin\alpha\right)\times \tag{19}$$
$$\left[\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha - \frac{1}{c^2}\mathbf{v}\times\left(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha\right)\right] \tag{20}$$
$$= q_e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) \tag{21}$$
$$= \mathbf{F} \tag{22}$$

To get the penultimate line it is just a matter of multiplying out the first expression and using $\cos^2\alpha + \sin^2\alpha = 1$, and cancelling off terms.

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