MAXWELL STRESS TENSOR: FORCE BETWEEN TWO CHARGES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.4.

Here's another, albeit impractical, example of using the Maxwell stress tensor to calculate the force on a charge. We have two charges q of the same sign, with one at $z=+a$ and the other at $z=-a$. We can integrate the stress tensor over some volume containing, say, the lower charge to find the force on it. (Obviously, we could just write down the force as $\mathbf{F}=-\frac{q}{4\pi\epsilon_0(2a)^2}\hat{\mathbf{z}}$ but we want to see if we can get the same answer from the stress tensor.)

We'll use a spherical coordinate system centred on the lower charge, and integrate the tensor over the xy plane (recall that we can do the integral over *any* volume enclosing the charge, and since the fields go to zero at infinity, we can use the lower half space as our volume, with the xy plane as its bounding surface). Then a point on the xy plane has rectangular coordinates $[r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta]$. Since the electric field radiates out from each of the charges with a magnitude

$$E=\frac{q}{4\pi\epsilon_0 r^2} \tag{1}$$

where r is the distance from the charge, the rectangular components of $\mathbf{E}_1$ from the lower charge are

$$E_{1x} = \frac{q}{4\pi\epsilon_0 r^2}\sin\theta\cos\phi \tag{2}$$

$$E_{1y} = \frac{q}{4\pi\epsilon_0 r^2}\sin\theta\sin\phi \tag{3}$$

$$E_{1z} = \frac{q}{4\pi\epsilon_0 r^2}\cos\theta \tag{4}$$

By symmetry, the x and y components of the upper charge's field are the same as those from the lower charge, and the z component is equal and opposite, so the total field in the xy plane is

$$E_x = \frac{2q}{4\pi\epsilon_0 r^2}\sin\theta\cos\phi \tag{5}$$
$$E_y = \frac{2q}{4\pi\epsilon_0 r^2}\sin\theta\sin\phi \tag{6}$$
$$E_z = 0 \tag{7}$$

Also from symmetry, the net force is in the z direction, as is the normal to the surface over which we're integrating, so we need only the component T_{zz}.

$$T_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) \tag{8}$$
$$= -\frac{\epsilon_0}{2}\left(\frac{q}{2\pi\epsilon_0 r^2}\sin\theta\right)^2 \tag{9}$$

Since we'll be integrating over the xy plane, we can use the 2-d polar coordinates s (for distance from the origin in the xy plane) and the azimuthal angle ϕ. We then have

$$r = \sqrt{a^2+s^2} \tag{10}$$
$$\sin\theta = \frac{s}{r} = \frac{s}{\sqrt{a^2+s^2}} \tag{11}$$
$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2+s^2}} \tag{12}$$
$$T_{zz} = -\frac{q^2}{8\pi^2\epsilon_0}\frac{s^2}{\left(a^2+s^2\right)^3} \tag{13}$$

The force is then given by

$$\mathbf{F} = \int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \tag{14}$$
$$= -\frac{q^2\hat{\mathbf{z}}}{8\pi^2\epsilon_0}(2\pi)\int_0^\infty \frac{s^2\,(s\,ds)}{\left(a^2+s^2\right)^3} \tag{15}$$

The integral can be done with software or by using the substitution $u = a^2+s^2$, $du = 2s\,ds$

$$\int_0^\infty \frac{s^3 ds}{(a^2+s^2)^3} = \int_{a^2}^\infty \frac{u-a^2}{2u^3} du \tag{16}$$

$$= \frac{1}{2a^2} - \frac{a^2}{4a^4} \tag{17}$$

$$= \frac{1}{4a^2} \tag{18}$$

Thus the force is

$$\mathbf{F} = -\frac{q^2}{4\pi\epsilon_0 (2a)^2}\hat{\mathbf{z}} \tag{19}$$

which is a roundabout way of getting Coulomb's law.

If the charges are opposite, then the x and y components of the field cancel and the z component adds, so we get for the force on the lower charge

$$E_x = 0 \tag{20}$$

$$E_y = 0 \tag{21}$$

$$E_z = \frac{2q}{4\pi\epsilon_0 r^2}\cos\theta \tag{22}$$

$$= \frac{qa}{2\pi\epsilon_0 (a^2+s^2)^{3/2}} \tag{23}$$

$$T_{zz} = \frac{q^2a^2}{8\pi^2\epsilon_0}\frac{1}{(a^2+s^2)^3} \tag{24}$$

$$\mathbf{F} = \frac{q^2a^2\hat{\mathbf{z}}}{8\pi^2\epsilon_0}(2\pi)\int_0^\infty \frac{s\,ds}{(a^2+s^2)^3} \tag{25}$$

$$= +\frac{q^2}{4\pi\epsilon_0 (2a)^2}\hat{\mathbf{z}} \tag{26}$$

This also agrees with Coulomb's law; this time the force is upwards so the charges are attracted to each other.