

## MAXWELL STRESS TENSOR: FORCE BETWEEN TWO CHARGES

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.4.

Here's another, albeit impractical, example of using the Maxwell stress tensor to calculate the force on a charge. We have two charges  $q$  of the same sign, with one at  $z = +a$  and the other at  $z = -a$ . We can integrate the stress tensor over some volume containing, say, the lower charge to find the force on it. (Obviously, we could just write down the force as  $\mathbf{F} = -\frac{q}{4\pi\epsilon_0(2a)^2}\hat{\mathbf{z}}$  but we want to see if we can get the same answer from the stress tensor.)

We'll use a spherical coordinate system centred on the lower charge, and integrate the tensor over the  $xy$  plane (recall that we can do the integral over *any* volume enclosing the charge, and since the fields go to zero at infinity, we can use the lower half space as our volume, with the  $xy$  plane as its bounding surface). Then a point on the  $xy$  plane has rectangular coordinates  $[r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta]$ . Since the electric field radiates out from each of the charges with a magnitude

$$(1) \quad E = \frac{q}{4\pi\epsilon_0 r^2}$$

where  $r$  is the distance from the charge, the rectangular components of  $\mathbf{E}_1$  from the lower charge are

$$(2) \quad E_{1x} = \frac{q}{4\pi\epsilon_0 r^2} \sin \theta \cos \phi$$

$$(3) \quad E_{1y} = \frac{q}{4\pi\epsilon_0 r^2} \sin \theta \sin \phi$$

$$(4) \quad E_{1z} = \frac{q}{4\pi\epsilon_0 r^2} \cos \theta$$

By symmetry, the  $x$  and  $y$  components of the upper charge's field are the same as those from the lower charge, and the  $z$  component is equal and opposite, so the total field in the  $xy$  plane is

$$(5) \quad E_x = \frac{2q}{4\pi\epsilon_0 r^2} \sin\theta \cos\phi$$

$$(6) \quad E_y = \frac{2q}{4\pi\epsilon_0 r^2} \sin\theta \sin\phi$$

$$(7) \quad E_z = 0$$

Also from symmetry, the net force is in the  $z$  direction, as is the normal to the surface over which we're integrating, so we need only the component  $T_{zz}$ .

$$(8) \quad T_{zz} = \frac{\epsilon_0}{2} (E_z^2 - E_x^2 - E_y^2)$$

$$(9) \quad = -\frac{\epsilon_0}{2} \left( \frac{q}{2\pi\epsilon_0 r^2} \sin\theta \right)^2$$

Since we'll be integrating over the  $xy$  plane, we can use the 2-d polar coordinates  $s$  (for distance from the origin in the  $xy$  plane) and the azimuthal angle  $\phi$ . We then have

$$(10) \quad r = \sqrt{a^2 + s^2}$$

$$(11) \quad \sin\theta = \frac{s}{r} = \frac{s}{\sqrt{a^2 + s^2}}$$

$$(12) \quad \cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2 + s^2}}$$

$$(13) \quad T_{zz} = -\frac{q^2}{8\pi^2\epsilon_0} \frac{s^2}{(a^2 + s^2)^3}$$

The force is then given by

$$(14) \quad \mathbf{F} = \int_{\mathcal{S}} \langle \mathbf{T} \rangle \cdot d\mathbf{a}$$

$$(15) \quad = -\frac{q^2 \hat{\mathbf{z}}}{8\pi^2\epsilon_0} (2\pi) \int_0^\infty \frac{s^2 (s ds)}{(a^2 + s^2)^3}$$

The integral can be done with software or by using the substitution  $u = a^2 + s^2$ ,  $du = 2s ds$

$$(16) \quad \int_0^\infty \frac{s^3 ds}{(a^2 + s^2)^3} = \int_{a^2}^\infty \frac{u - a^2}{2u^3} du$$

$$(17) \quad = \frac{1}{2a^2} - \frac{a^2}{4a^4}$$

$$(18) \quad = \frac{1}{4a^2}$$

Thus the force is

$$(19) \quad \mathbf{F} = -\frac{q^2}{4\pi\epsilon_0(2a)^2}\hat{\mathbf{z}}$$

which is a roundabout way of getting Coulomb's law.

If the charges are opposite, then the  $x$  and  $y$  components of the field cancel and the  $z$  component adds, so we get for the force on the lower charge

$$(20) \quad E_x = 0$$

$$(21) \quad E_y = 0$$

$$(22) \quad E_z = \frac{2q}{4\pi\epsilon_0 r^2} \cos \theta$$

$$(23) \quad = \frac{qa}{2\pi\epsilon_0 (a^2 + s^2)^{3/2}}$$

$$(24) \quad T_{zz} = \frac{q^2 a^2}{8\pi^2 \epsilon_0} \frac{1}{(a^2 + s^2)^3}$$

$$(25) \quad \mathbf{F} = \frac{q^2 a^2 \hat{\mathbf{z}}}{8\pi^2 \epsilon_0} (2\pi) \int_0^\infty \frac{s ds}{(a^2 + s^2)^3}$$

$$(26) \quad = +\frac{q^2}{4\pi\epsilon_0(2a)^2}\hat{\mathbf{z}}$$

This also agrees with Coulomb's law; this time the force is upwards so the charges are attracted to each other.