MOMENTUM IN ELECTROMAGNETIC FIELDS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.5.

We've seen that the force on a collection of charges in a volume $\mathcal{V}$ enclosed within a surface S due to electromagnetic fields is given in terms of the Maxwell stress tensor:

$$\mathbf{F}=\int_{\mathcal{S}}\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}-\varepsilon_0\mu_0\frac{\partial}{\partial t}\int_{\mathcal{V}}\mathbf{S}d^3\mathbf{r} \tag{1}$$

where $\overleftrightarrow{\mathbf{T}}$ is the stress tensor and $\mathbf{S}$ is the Poynting vector. It turns out that electromagnetic fields carry both momentum and angular momentum. To see this, we can write the above equation in terms of the mechanical momentum $\mathbf{p}_{mech}$ (that is, the momentum of the particles present, rather than the fields):

$$\frac{d\mathbf{p}_{mech}}{dt}=\int_{\mathcal{S}}\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}-\varepsilon_0\mu_0\frac{\partial}{\partial t}\int_{\mathcal{V}}\mathbf{S}d^3\mathbf{r} \tag{2}$$

Looking back at Poynting's theorem:

$$\frac{dW}{dt}=-\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}}\left(\frac{1}{\mu_0}B^2+\varepsilon_0E^2\right)d^3\mathbf{r}-\int_{\mathcal{S}}\mathbf{S}\cdot d\mathbf{a} \tag{3}$$

we saw that there, we had a volume integral representing the rate at which energy in the fields was changing and a surface integral representing the rate at which energy was flowing into (or out of) the volume. If we make an analogy with 2, we can interpret the volume integral as the rate at which momentum in the fields is changing and the surface integral as the rate at which momentum flows into the volume. That is

$$\mathbf{p}_{em}\equiv\varepsilon_0\mu_0\int_{\mathcal{V}}\mathbf{S}d^3\mathbf{r} \tag{4}$$

is the momentum of the fields themselves and

$$\mathfrak{p}_{em}=\varepsilon_0\mu_0\mathbf{S} \tag{5}$$

is the momentum density of the fields.

The other integral can be converted into a volume integral using the divergence theorem:

$$\int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = \int_{\mathcal{V}} \nabla \cdot \overleftrightarrow{\mathbf{T}} d^3\mathbf{r} \tag{6}$$

If we define the mechanical momentum density $\mathfrak{p}_{mech}$ then we can write 2 in differential form

$$\frac{\partial}{\partial t}\left(\mathfrak{p}_{em} + \mathfrak{p}_{mech}\right) = \nabla \cdot \overleftrightarrow{\mathbf{T}} \tag{7}$$

We can look at this equation as a set of three equations, one for each component of the momentum. That is,

$$\frac{\partial}{\partial t}\left(\mathfrak{p}_{em} + \mathfrak{p}_{mech}\right)_i = \sum_{j=1}^{3} \partial_j T_{ij} \tag{8}$$

Compare this with the continuity equation relating charge density ρ and current density $\mathbf{J}$:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \tag{9}$$

This says that if there is a divergence in $\mathbf{J}$, that is, current is flowing out of some infinitesimal volume, then the charge density decreases in that volume. Equation 7 says something similar by relating the total momentum density $\mathfrak{p}_{em} + \mathfrak{p}_{mech}$ to the divergence of the stress tensor $\nabla \cdot \overleftrightarrow{\mathbf{T}}$. In particular, the rate at which a particular component i of momentum changes is the divergence of the ith column of $-\overleftrightarrow{\mathbf{T}}$ (the minus sign occurs because a positive divergence of $\overleftrightarrow{\mathbf{T}}$ leads to an *increase* in momentum density, so it must be the negative of $\overleftrightarrow{\mathbf{T}}$ that gives the rate at which momentum flows *out* of a volume), so $-T_{ij}$ is the rate at which momentum component i flows in the j direction. This is the same interpretation as the spatial coordinates of the stress-energy tensor in relativity.

In electrostatics, $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = 0$ because $\mathbf{B} = 0$, so electric fields on their own store no momentum. However, the stress tensor is not zero in electrostatics, so momentum still does flow in an electric field.

Example. For example, consider an infinite flat plate capacitor with surface charge density $+\sigma$ on one plate and $-\sigma$ on the other. From Gauss's law, the electric field from each plate is

$$E = \frac{\sigma}{2\varepsilon_0} \tag{10}$$

pointing away from the plate on both sides. Outside the capacitor, the fields cancel, but between the plates they add, giving a net field of

$$E = \frac{\sigma}{\varepsilon_0} \tag{11}$$

Taking the z axis to be perpendicular to the plates, we have for the stress tensor between the plates

$$\overleftrightarrow{\mathbf{T}} = \frac{\varepsilon_0}{2}\begin{bmatrix} -E_z^2 & 0 & 0 \\ 0 & -E_z^2 & 0 \\ 0 & 0 & E_z^2 \end{bmatrix} \tag{12}$$

$$= \frac{\sigma^2}{2\varepsilon_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

We can get the force on a plate from 1. Since $d\mathbf{a}$ is in the $-z$ direction, only T_{zz} contributes (and $\mathbf{S} = 0$ so the second integral is zero), so if we integrate over a unit area of the plate

$$\mathbf{F}_{upper} = \int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \tag{14}$$

$$= -T_{zz}\hat{\mathbf{z}} \tag{15}$$

$$= -\frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}} \tag{16}$$

The minus sign occurs because $d\mathbf{a}$ points in the $-z$ direction and T_{zz} is positive. On the lower plate, $d\mathbf{a}$ points in the $+z$ direction and T_{zz} is the same, so $\mathbf{F}_{lower} = +\frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}}$. The force on each plate is towards the other plate, as we'd expect since they are oppositely charged.

Interpreting the stress tensor as momentum flux density, we have that $-T_{zz}$ is the momentum flux in the z direction; that is, it's the amount of momentum per unit area per unit time that crosses any plane parallel to and between the plates. When this momentum is absorbed by a plate, the plate experiences a force. The momentum flux density flowing in the $+z$ direction is therefore $-T_{zz} = -\frac{\sigma^2}{2\varepsilon_0}$ and this is the momentum delivered to the plate per unit area per unit time, which is the force per unit area, so we get the same answer as in 16.

To get the force on the lower plate, we can use the fact that a momentum flux density of $-T_{zz}$ in the $+z$ direction is the same as a density of $+T_{zz}$ in the $-z$ direction, so the momentum delivered to the lower plate per unit area per unit time is $+T_{zz}=+\frac{\sigma^2}{2\varepsilon_0}$ giving $\mathbf{F}_{lower}=+\frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}}$ as before.

Incidentally, it might seem that equations 1 and 7 are inconsistent in this case where $\overleftrightarrow{\mathbf{T}}$ is constant, since it would appear that $\nabla\cdot\overleftrightarrow{\mathbf{T}}=0$ which would imply, by the divergence theorem, that $\int_{\mathcal{S}}\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}=0$ as well. The catch is that $\overleftrightarrow{\mathbf{T}}$ isn't really constant over the surface surrounding a unit area on one of the plates, since it has non-zero components between the plates, but is zero everywhere outside the plates. Thus $\overleftrightarrow{\mathbf{T}}$ must drop to zero as we cross the plate. In an idealized problem where the plate is infinitesimally thin, we could use a step function to represent $\overleftrightarrow{\mathbf{T}}$, which means that $\nabla\cdot\overleftrightarrow{\mathbf{T}}$ would involve delta functions.

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