MOMENTUM IN A CAPACITOR

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.6.

The momentum density of an electromagnetic field is given by

$$\mathfrak{p}_{em} = \epsilon_0 \mu_0 \mathbf{S} = \epsilon_0 \mathbf{E} \times \mathbf{B} \tag{1}$$

Rather bizarrely, this applies even in cases where the electric and magnetic fields do not change with time, so a system such as a coaxial cable carrying a steady current, with a potential difference between its inner and outer cylinders, contains momentum in the fields. In fact, there is a balancing momentum due to a relativistic effect involving magnetic dipoles which balances the momentum in the fields so that a static configuration actually does have zero momentum, but that takes us further afield than we want to go at the moment.

As an example of conservation of the field momentum, we can look at an idealized parallel plate capacitor, with plate area A and plate separation d. Its capacitance is given by

$$C = \epsilon_0 \frac{A}{d} \tag{2}$$

If we align the plates parallel to the xy plane and charge the capacitor, then the electric field between the plates is $\mathbf{E} = E_0 \hat{\mathbf{z}}$ assuming that the lower plate is positive and the upper one negative. If the potential difference between the plates is V_0 then $E_0 = V_0/d$.

We now apply a constant magnetic field $\mathbf{B} = B_0 \hat{\mathbf{x}}$ along the $+x$ direction, so that the momentum density between the plates (neglecting edge effects) is

$$\mathfrak{p}_{em} = \epsilon_0 E_0 B_0 \hat{\mathbf{y}} \tag{3}$$

and the total momentem in the fields between the plates is just the density multiplied by the volume:

$$\mathbf{p}_{em} = \epsilon_0 E_0 B_0 A d \hat{\mathbf{y}} \tag{4}$$

Now suppose we connect a resistor R between the plates along the z axis. The capacitor will discharge, with the potential difference given by

$$V(t) = V_0 e^{-t/RC} \tag{5}$$

The current in the resistor is in the $+z$ direction of magnitude

$$I(t) = \frac{V_0}{R} e^{-t/RC} \tag{6}$$

Because we now have moving charges (the current) in a magnetic field, the resistor feels a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = dI(t) B\hat{\mathbf{y}} \tag{7}$$

The total impulse $\mathbf{I}$ felt by the resistor as the capacitor discharges (the impulse is the force integrated over time, and gives the change in momentum a force imparts to an object) is

$$\begin{aligned}
\mathbf{I} &= \int_0^\infty \mathbf{F} dt && (8)\\
&= \frac{dV_0 B}{R} \hat{\mathbf{y}} \int_0^\infty e^{-t/RC} dt && (9)\\
&= C d V_0 B \hat{\mathbf{y}} && (10)\\
&= \epsilon_0 \frac{A}{d} d E_0 d B \hat{\mathbf{y}} && (11)\\
&= \epsilon_0 E_0 B_0 A d \hat{\mathbf{y}} && (12)\\
&= \mathbf{p}_{em} && (13)
\end{aligned}$$

Thus the momentum in the fields gets transferred to the resistor after the capacitor discharges.

We can also eliminate the momentum in the field by reducing the magnetic field to zero and leaving the capacitor charged. By Faraday's law, the changing magnetic field will induce an electric field (in addition to the field due to the charge on the plates) by the formula

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{14}$$

or in integral form

$$\oint \mathbf{E} \cdot d\ell = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{15}$$

We can choose the path of integration on the LHS to be a rectangle parallel to the yz plane that is a cross-section of the area between the plates. If the width of the plates is ℓ then this rectangle has width ℓ and height d. If we reduce **B** uniformly, then at any particular time $\frac{\partial \mathbf{B}}{\partial t}$ is a constant over the rectangle and the RHS can be evaluated so we get

$$\oint \mathbf{E}\cdot d\ell = -\frac{\partial B}{\partial t}\ell d \tag{16}$$

As for **E**, remember that we're considering only the field that is induced due to changing **B**. **E** cannot have a z component since, if it did, we could invert this z component by turning the system upside-down. However, since **B** is uniform and points in the x direction, turning it upside-down would leave it unchanged, so the induced **E** cannot change either, so $E_z = 0$. We must also have $E_x = 0$ since any path in a plane parallel to the x axis contains no magnetic flux (since it's parallel to **B**; the same argument also shows that $E_z = 0$). Thus the induced field must be entirely in the y direction.

If we neglect edge effects in the capacitor, then by symmetry**E** is constant over any plane parallel to the plates, so the integral on the LHS becomes (taking the path of integration as counterclockwise when viewed down the x axis)

$$\oint \mathbf{E}\cdot d\ell = -\ell E_y(z=d) + \ell E_y(z=0) \tag{17}$$

$$E(0) - E(d) = -\frac{\partial B}{\partial t}d \tag{18}$$

This field exerts a force in the y direction on the plates. If the surface charge density on the lower plate is $+\sigma$ and on the upper plate is $-\sigma$ then the net force on the capacitor is

$$\mathbf{F} = \sigma A\left[E(0) - E(d)\right]\hat{\mathbf{y}} \tag{19}$$

$$= -\sigma A\frac{\partial B}{\partial t}d\hat{\mathbf{y}} \tag{20}$$

The impulse is then

$$\mathbf{I} = -\sigma A d\hat{\mathbf{y}}\int_0^\infty \frac{\partial B}{\partial t}dt \tag{21}$$

$$= \sigma A d B_0 \hat{\mathbf{y}} \tag{22}$$

(The minus sign is cancelled out because B is decreasing so $\frac{\partial B}{\partial t} < 0$.) The surface charge density on a capacitor plate is related to the vertical electric field by

$$E_0 = \frac{\sigma}{\epsilon_0} \tag{23}$$

so the impulse is

$$\mathbf{I} = \epsilon_0 E_0 A d B_0 \hat{\mathbf{y}} \tag{24}$$

which is the same as 12. Thus the field's momentum is transferred to the capacitor as we turn off the magnetic field.