ANGULAR MOMENTUM IN ELECTROMAGNETIC FIELDS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.7.

The momentum density of an electromagnetic field is given by

$$\mathfrak{p}_{em} = \varepsilon_0\mu_0\mathbf{S} = \varepsilon_0\mathbf{E}\times\mathbf{B} \tag{1}$$

If we have linear momentum, then we automatically have *angular momentum* with respect to some origin by using the classical definition of angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. We can define the angular momentum density of an electromagnetic field by

$$\mathfrak{L}_{em} \equiv \mathbf{r}\times\mathfrak{p}_{em} = \varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) \tag{2}$$

Just as with linear momentum, even static fields can have angular momentum. As an example, suppose we have a long solenoid with n turns per unit length carrying current I_0 and a radius R, with its axis along the z axis. The magnetic field inside the solenoid is

$$\mathbf{B}_0 = \mu_0 nI\hat{\mathbf{z}} \tag{3}$$

The field is zero outside the solenoid.

Now suppose we add two other cylinders (not solenoids), both coaxial with the solenoid. One cylinder has radius $a < R$ (so it lies inside the solenoid) and carries surface charge $+Q$; the other cylinder has radius $b > R$ (outside the solenoid) and carries charge $-Q$. Both cylinders have length ℓ. From Gauss's law, the electric field between these two cylinders is, for $a < r < b$

$$\mathbf{E}_0 = \frac{Q}{2\pi\varepsilon_0\ell}\frac{\hat{\mathbf{r}}}{r} \tag{4}$$

That is, the field points radially outward from the axis. The electric field is zero for $r < a$ and $r > b$. (We're neglecting end effects, so we're assuming that $\ell \gg b > a$.)

The linear momentum density is non-zero in the region $a < r < R$ (where both fields are non-zero) and we have

$$\mathfrak{p}_{em} = -\frac{\mu_0 nIQ}{2\pi\ell}\frac{\hat{\phi}}{r} \tag{5}$$

so the angular momentum density is

$$\begin{aligned}\mathfrak{L}_{em} &= \mathbf{r}\times\mathfrak{p}_{em} &(6)\\ &= -\frac{\mu_0 nIQ}{2\pi\ell}\hat{\mathbf{z}} &(7)\end{aligned}$$

Conveniently, this is constant so the total angular momentum is just the density times the volume of the cylindrical tube in the region $a<r<R$

$$\begin{aligned}\mathbf{L}_{em} &= -\ell\pi\left(R^2-a^2\right)\frac{\mu_0 nIQ}{2\pi\ell}\hat{\mathbf{z}} &(8)\\ &= -\left(R^2-a^2\right)\frac{\mu_0 nIQ}{2}\hat{\mathbf{z}} &(9)\end{aligned}$$

Now suppose we (quasistatically) discharge the two cylinders by connecting a resistor $\mathcal{R}$ between them. We'd like to show that the angular momentum gets transferred from the fields to the physical devices in the problem. The two cylinders are effectively a capacitor with some capacitance C, so we know that the current in the resistor will decay exponentially

$$I(t) = \frac{V_0}{\mathcal{R}}e^{-t/\mathcal{R}C} \tag{10}$$

where V_0 is the potential difference between the cylinders at $t=0$. The force $d\mathbf{F}$ on a segment of the resistor of length dr is

$$\begin{aligned}d\mathbf{F} &= I(t)\,dr\hat{\mathbf{r}}\times\mathbf{B}_0 &(11)\\ &= -I(t)\,drB_0\hat{\phi} &(12)\end{aligned}$$

so the torque on this segment is

$$\begin{aligned}d\mathbf{N} &= \mathbf{r}\times d\mathbf{F} &(13)\\ &= -I(t)\,B_0 r dr\hat{\mathbf{z}} &(14)\end{aligned}$$

The total torque on the resistor at time t is

$$\mathbf{N}(t) = -I(t)B_0\hat{\mathbf{z}}\int_a^R r\,dr \tag{15}$$

$$= -\frac{1}{2}I(t)B_0\left(R^2-a^2\right)\hat{\mathbf{z}} \tag{16}$$

The angular impulse is the integral of torque over time, so we get

$$\mathbf{I} = \int_0^\infty \mathbf{N}(t)\,dt \tag{17}$$

$$= -\frac{1}{2}B_0\left(R^2-a^2\right)\hat{\mathbf{z}}\int_0^\infty I(t)\,dt \tag{18}$$

$$= -\frac{1}{2}B_0\left(R^2-a^2\right)\hat{\mathbf{z}}\int_0^\infty \frac{V_0}{\mathcal{R}}e^{-t/\mathcal{R}C}dt \tag{19}$$

$$= -\frac{1}{2}B_0\left(R^2-a^2\right)CV_0\hat{\mathbf{z}} \tag{20}$$

$$= -\frac{1}{2}\mu_0 nI\left(R^2-a^2\right)Q\hat{\mathbf{z}} \tag{21}$$

where we used the relation between capacitance, charge and voltage $Q = CV$. We see that this agrees with 9, so all the angular momentum in the fields is transferred to the resistor as the electric field is reduced to zero.

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