ANGULAR MOMENTUM IN A MAGNETIZED SPHERE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.8.

Here's another slightly more complex example of the conservation of angular momentum in electromagnetic fields. We have a solid iron sphere of radius R with a uniform magnetization $M\hat{\mathbf{z}}$ and carrying a surface charge Q. We want to show that the angular momentem in the electric and magnetic fields is conserved as we switch off either the magnetic or electric field.

First, we need to calculate the angular momentum in the fields before anything is switched off.

The momentum density of an electromagnetic field is given by

$$\mathfrak{p}_{em}=\varepsilon_0\mu_0\mathbf{S}=\varepsilon_0\mathbf{E}\times\mathbf{B} \tag{1}$$

and the angular momentum density is

$$\mathfrak{L}_{em}\equiv\mathbf{r}\times\mathfrak{p}_{em}=\varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) \tag{2}$$

Griffiths shows in his Example 6.1 that the magnetic field of a uniformly magnetized sphere is

$$\mathbf{B}=\begin{cases}\frac{2}{3}\mu_0M\hat{\mathbf{z}}=\frac{2}{3}\mu_0M\left(\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}\right) & \text{inside}\\ \frac{\mu_0MR^3}{3r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right) & \text{outside}\end{cases} \tag{3}$$

The outside formula is just the field of a perfect magnetic dipole.

The electric field is

$$\mathbf{E}=\begin{cases}0 & \text{inside}\\ \frac{Q}{4\pi\varepsilon_0r^2}\hat{\mathbf{r}} & \text{outside}\end{cases} \tag{4}$$

The linear momentum density is then

$$\mathfrak{p}_{em}=\varepsilon_0\mathbf{E}\times\mathbf{B}=\frac{\mu_0MQR^3}{12\pi r^5}\sin\theta\hat{\boldsymbol{\phi}} \tag{5}$$

so the angular momentum density is

$$\mathfrak{L}_{em} = \mathbf{r} \times \mathfrak{p}_{em} = -\frac{\mu_0 M Q R^3}{12\pi r^4} \sin\theta\,\hat{\theta} \tag{6}$$

The total angular momentum is thus

$$\mathbf{L}_{em} = -\frac{\mu_0 M Q R^3}{12\pi} \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{\sin\theta}{r^4} r^2 \sin\theta\,\hat{\theta}\,d\phi\,d\theta\,dr \tag{7}$$

As the unit vector $\hat{\theta}$ is not constant in direction, we need to express it in rectangular coordinates:

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \tag{8}$$

The integral of $\cos\phi$ and $\sin\phi$ over the interval $[0, 2\pi]$ gives zero, so only the z component survives, giving

$$\mathbf{L}_{em} = \frac{\mu_0 M Q R^3}{12\pi} \hat{\mathbf{z}}\,(2\pi) \int_R^\infty \frac{dr}{r^2} \int_0^\pi \sin^3\theta\,d\theta \tag{9}$$

$$= \frac{2}{9} \mu_0 M Q R^2 \hat{\mathbf{z}} \tag{10}$$

Now suppose we slowly switch off the magnetic field, keeping Q constant. This will generate an electric field according to Faraday's law. Consider a slice of the sphere parallel to the xy plane, that is, at some value of θ. The electric field is circular around the axis of the sphere and satisfies the equation

$$\oint \mathbf{E} \cdot d\ell = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{11}$$

The radius of the circle is $R\sin\theta$ and if we switch off the magnetic field in a uniform way, then $\frac{\partial \mathbf{B}}{\partial t}$ is constant over this area, so we get

$$2\pi R E \sin\theta = -\pi (R\sin\theta)^2 \frac{\partial B}{\partial t} \tag{12}$$

$$\mathbf{E} = -\frac{1}{2} R \sin\theta \frac{\partial B}{\partial t} \hat{\phi} \tag{13}$$

The minus sign in the last line cancels the fact that $\frac{\partial B}{\partial t} < 0$ and gives an electric field that opposes the decrease in magnetic field.

This field exerts a force on the strip of charge in the slice. The surface charge density σ is

$$\sigma = \frac{Q}{4\pi R^2} \tag{14}$$

so the amount of charge in the slice is

$$dq = \frac{Q}{4\pi R^2}\left(2\pi R\sin\theta\right)\left(Rd\theta\right) = \frac{1}{2}Q\sin\theta d\theta \tag{15}$$

The force on the slice is

$$d\mathbf{F} = \mathbf{E}dq = -\frac{1}{4}RQ\sin^2\theta\, d\theta\frac{\partial B}{\partial t}\hat{\phi} \tag{16}$$

and the torque is

(17)

$$d\mathbf{N} = \mathbf{r}\times d\mathbf{F} = -\left(R\sin\theta\right)\frac{1}{4}RQ\sin^2\theta\, d\theta\frac{\partial B}{\partial t}\hat{\mathbf{z}} = -\frac{1}{4}R^2Q\sin^3\theta\, d\theta\frac{\partial B}{\partial t}\hat{\mathbf{z}}$$

The total torque on the sphere is

$$\begin{aligned}
\mathbf{N} &= \int_0^{\pi} d\mathbf{N}\left(\theta\right) && (18)\\
&= -\frac{1}{4}R^2Q\frac{\partial B}{\partial t}\hat{\mathbf{z}}\int_0^{\pi}\sin^3\theta d\theta && (19)\\
&= -\frac{1}{3}QR^2\frac{\partial B}{\partial t}\hat{\mathbf{z}} && (20)
\end{aligned}$$

The impulse is obtained by integrating this over time:

$$\mathbf{I} = -\frac{1}{3}QR^2\hat{\mathbf{z}}\int_0^{\infty}\frac{\partial B}{\partial t}dt = -\frac{1}{3}QR^2\hat{\mathbf{z}}\left(-\frac{2}{3}\mu_0 M\right) = \frac{2}{9}\mu_0 MQR^2\hat{\mathbf{z}} \tag{21}$$

which agrees with 10, showing that the angular momentum is transferred to the sphere.

Now let's keep the magnetization constant and decrease the electric field by draining off the sphere's surface charge. We arrange to do this by connecting a resistor $\mathscr{R}$ between the north pole and ground, but we do this in a way that the surface charge density remains uniform over the sphere as the total charge decreases. Doing this creates a surface current that experiences a force from the magnetic field. Since we're draining the charge off at the north pole, the surface current everywhere on the sphere will head towards $\theta = 0$, that is, it flows in the $-\theta$ direction. To work out the current at each point on the sphere, consider the same horizontal slice that we used above. The current passing through this slice must be the sum of the current passing through the slice just below it (that is, at $\theta + d\theta$) and the charge draining away from the slice itself. That is

$$I(\theta) = I(\theta + d\theta) + \frac{d\sigma}{dt}(2\pi R \sin\theta)(R d\theta) \tag{22}$$

$$\frac{dI}{d\theta} = -\frac{d\sigma}{dt} 2\pi R^2 \sin\theta \tag{23}$$

$$I(\theta) = \frac{d\sigma}{dt} 2\pi R^2 \cos\theta + \alpha \tag{24}$$

where α is a constant of integration. Since the charge is draining off through the north pole, we'd like the current at the south pole to be zero, so we choose

$$\alpha = \frac{d\sigma}{dt} 2\pi R^2 \tag{25}$$

We can treat the sphere/ground system as a capacitor C, so the charge density is given by

$$\sigma(t) = \sigma(0) e^{-t/\mathscr{R}C} \tag{26}$$

$$= \frac{Q}{4\pi R^2} e^{-t/\mathscr{R}C} \tag{27}$$

so

$$I(\theta) = -\frac{Q}{2\mathscr{R}C} e^{-t/\mathscr{R}C}(1 + \cos\theta) \tag{28}$$

This is the total current flowing across the surface of a slice at angle θ. The $q\mathbf{v}$ term in the force law is the current times the distance over which it flows, which in this case is $R d\theta$. We can now apply the Lorentz force law $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ to work out the force on the sphere due to the current interacting with the magnetic field. Since the current is parallel to $\hat{\theta}$, only the r component of $\mathbf{B}$ will survive the cross product. Note that although the θ component of $\mathbf{B}$ is not continuous across the surface of the sphere, the r component *is* continuous, so it doesn't matter whether we use the inside or outside formula. Thus we get

$$d\mathbf{F} = \left(-\frac{Q}{2\mathscr{R}C}e^{-t/\mathscr{R}C}\left(1+\cos\theta\right)\right)\left(Rd\theta\right)\left(\frac{2\mu_0 M}{3}\cos\theta\right)\left(-\hat{\phi}\right) \tag{29}$$

$$= \frac{\mu_0 MQR}{3\mathscr{R}C}e^{-t/\mathscr{R}C}\hat{\phi}\left(\cos\theta+\cos^2\theta\right) \tag{30}$$

The torque on a slice is

$$d\mathbf{N} = \mathbf{r}\times d\mathbf{F} \tag{31}$$

$$= \frac{\mu_0 MQR^2}{3\mathscr{R}C}e^{-t/\mathscr{R}C}\hat{\mathbf{z}}\left(\cos\theta+\cos^2\theta\right)\sin\theta \tag{32}$$

The total torque is thus

$$\mathbf{N} = \frac{\mu_0 MQR^2}{3\mathscr{R}C}e^{-t/\mathscr{R}C}\hat{\mathbf{z}}\int_0^{\pi}\left(\cos\theta+\cos^2\theta\right)\sin\theta d\theta \tag{33}$$

$$= \frac{2\mu_0 MQR^2}{9\mathscr{R}C}e^{-t/\mathscr{R}C}\hat{\mathbf{z}} \tag{34}$$

Finally we integrate the torque over time to get the impulse:

$$\mathbf{I} = \int_0^{\infty}\mathbf{N}dt \tag{35}$$

$$= \frac{2\mu_0 MQR^2}{9\mathscr{R}C}\hat{\mathbf{z}}\int_0^{\infty}e^{-t/\mathscr{R}C}dt \tag{36}$$

$$= \frac{2}{9}\mu_0 MQR^2\hat{\mathbf{z}} \tag{37}$$

This again agrees with 10 so all is well.