ENERGY TRANSFER IN A SOLENOID

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.9.

Here's a simple example of conservation of energy in an electromagnetic system. We have an infinite solenoid of radius a carrying n turns per unit length and current I_s. The magnetic field is zero outside the solenoid and inside we have

$$\mathbf{B}_s = \mu_0 n I_s \hat{\mathbf{z}} \tag{0.1}$$

Now suppose we put a circular wire loop of radius $b \gg a$ and resistance R around the solenoid. If we now decrease the current in the solenoid, the changing magnetic flux will induce a circumferential electric field around the solenoid, which will in turn create a current in the wire. To find the current I_r in the resistor we can use the fact that the electric field creates an electromotive force (emf) around the wire:

$$\mathscr{E} = -\frac{d\Phi}{dt} \tag{0.2}$$

$$= -\pi a^2 \mu_0 n \frac{dI_s}{dt} \tag{0.3}$$

The current is

$$I_r = \frac{\mathscr{E}}{R} = -\frac{\pi a^2 \mu_0 n}{R}\frac{dI_s}{dt} \tag{0.4}$$

To find the direction of the current, remember that it in turn generates a magnetic field that opposes the reduction in the solenoid's field, so the current must flow in the $+\phi$ direction (counterclockwise as viewed from above).

The rate at which energy is dissipated by the resistor is the power, which is $I_r^2 R$. This energy must come from the solenoid via the Poynting vector. We can calculate the Poynting vector just outside the solenoid as follows. First, we need $\mathbf{E}$ and $\mathbf{B}$ outside the solenoid. The electric field is produced by the changing magnetic field in the solenoid, and by Faraday's law we have

$$\oint \mathbf{E}\cdot d\ell = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} \tag{0.5}$$

Taking a circular path of radius a we get

$$2\pi a E = -\mu_0 n \frac{dI_s}{dt}\pi a^2 \tag{0.6}$$

$$\mathbf{E} = -\frac{\mu_0 n a}{2}\frac{dI_s}{dt}\hat{\phi} \tag{0.7}$$

$$= \frac{I_r R}{2\pi a}\hat{\phi} \tag{0.8}$$

Remember that $\frac{dI_s}{dt} < 0$ so $\mathbf{E}$ points in the $+\phi$ direction.

There is no magnetic field due to the solenoid outside the solenoid itself, but the current in the resistor generates a magnetic field due to the Biot-Savart law. Griffiths works out the magnetic field on the z axis due to a circular loop in his Example 5.6 and since we're taking the radius b of the loop to be much greater than the radius a of the solenoid, we can use this formula as a good approximation. We have

$$\mathbf{B}_r = \frac{\mu_0 I_r}{2}\frac{b^2}{\left(b^2+z^2\right)^{3/2}}\hat{\mathbf{z}} \tag{0.9}$$

The Poynting vector is then

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{0.10}$$

$$= \frac{I_r^2 R}{4\pi a}\frac{b^2}{\left(b^2+z^2\right)^{3/2}}\hat{\mathbf{r}} \tag{0.11}$$

We can integrate the magnitude of the vector over the surface of the solenoid to find the rate at which energy is radiating away from the solenoid. We get

$$P = \int \mathbf{S}\cdot d\mathbf{a} \tag{0.12}$$

$$= (2\pi a)\frac{I_r^2 R}{4\pi a}\int_{-\infty}^{\infty}\frac{b^2}{\left(b^2+z^2\right)^{3/2}}dz \tag{0.13}$$

$$= I_r^2 R \tag{0.14}$$

Thus the power in the resistor is indeed coming from the solenoid.