

ENERGY TRANSFER IN A SOLENOID

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.9.

Here's a simple example of conservation of energy in an electromagnetic system. We have an infinite solenoid of radius a carrying n turns per unit length and current I_s . The magnetic field is zero outside the solenoid and inside we have

$$\mathbf{B}_s = \mu_0 n I_s \hat{\mathbf{z}} \quad (1)$$

Now suppose we put a circular wire loop of radius $b \gg a$ and resistance R around the solenoid. If we now decrease the current in the solenoid, the changing magnetic flux will induce a circumferential electric field around the solenoid, which will in turn create a current in the wire. To find the current I_r in the resistor we can use the fact that the electric field creates an electromotive force (emf) around the wire:

$$\mathcal{E} = -\frac{d\Phi}{dt} \quad (2)$$

$$= -\pi a^2 \mu_0 n \frac{dI_s}{dt} \quad (3)$$

The current is

$$I_r = \frac{\mathcal{E}}{R} = -\frac{\pi a^2 \mu_0 n}{R} \frac{dI_s}{dt} \quad (4)$$

To find the direction of the current, remember that it in turn generates a magnetic field that opposes the reduction in the solenoid's field, so the current must flow in the $+\phi$ direction (counterclockwise as viewed from above).

The rate at which energy is dissipated by the resistor is the power, which is $I_r^2 R$. This energy must come from the solenoid via the Poynting vector. We can calculate the Poynting vector just outside the solenoid as follows. First, we need \mathbf{E} and \mathbf{B} outside the solenoid. The electric field is produced by the changing magnetic field in the solenoid, and by Faraday's law we have

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \quad (5)$$

Taking a circular path of radius a we get

$$2\pi a E = -\mu_0 n \frac{dI_s}{dt} \pi a^2 \quad (6)$$

$$\mathbf{E} = -\frac{\mu_0 n a}{2} \frac{dI_s}{dt} \hat{\phi} \quad (7)$$

$$= \frac{I_r R}{2\pi a} \hat{\phi} \quad (8)$$

Remember that $\frac{dI_s}{dt} < 0$ so \mathbf{E} points in the $+\hat{\phi}$ direction.

There is no magnetic field due to the solenoid outside the solenoid itself, but the current in the resistor generates a magnetic field due to the Biot-Savart law. Griffiths works out the magnetic field on the z axis due to a circular loop in his Example 5.6 and since we're taking the radius b of the loop to be much greater than the radius a of the solenoid, we can use this formula as a good approximation. We have

$$\mathbf{B}_r = \frac{\mu_0 I_r}{2} \frac{b^2}{(b^2 + z^2)^{3/2}} \hat{\mathbf{z}} \quad (9)$$

The Poynting vector is then

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (10)$$

$$= \frac{I_r^2 R}{4\pi a} \frac{b^2}{(b^2 + z^2)^{3/2}} \hat{\mathbf{r}} \quad (11)$$

We can integrate the magnitude of the vector over the surface of the solenoid to find the rate at which energy is radiating away from the solenoid. We get

$$P = \int \mathbf{S} \cdot d\mathbf{a} \quad (12)$$

$$= (2\pi a) \frac{I_r^2 R}{4\pi a} \int_{-\infty}^{\infty} \frac{b^2}{(b^2 + z^2)^{3/2}} dz \quad (13)$$

$$= I_r^2 R \quad (14)$$

Thus the power in the resistor is indeed coming from the solenoid.