

MOMENTUM IN A MAGNETIZED AND POLARIZED SPHERE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.10.

Here's another example of finding momentum in electromagnetic fields. Consider a non-conducting sphere of radius R with a uniform magnetization \mathbf{M} and a uniform polarization \mathbf{P} , which need not be in the same direction as \mathbf{M} . What is the total momentum stored in the fields?

The momentum density is

$$(0.1) \quad \mathbf{p}_{em} = \epsilon_0 \mathbf{E} \times \mathbf{B}$$

It's easiest to work with the fields in their coordinate free forms. The field of a pure magnetic dipole at the origin is

$$(0.2) \quad \mathbf{B}_1 = \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}]$$

and for a pure electric dipole we have

$$(0.3) \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}]$$

Griffiths shows in his Example 6.1 that the magnetic field of a uniformly magnetized sphere is

$$(0.4) \quad \mathbf{B} = \begin{cases} \frac{2}{3}\mu_0\mathbf{M} & \text{inside} \\ \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}] & \text{outside} \end{cases}$$

where the total magnetic dipole moment of the sphere is

$$(0.5) \quad \mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$$

so

$$(0.6) \quad \mathbf{B} = \begin{cases} \frac{2}{3}\mu_0\mathbf{M} & \text{inside} \\ \frac{\mu_0 R^3}{r^3} [(\mathbf{M} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \frac{1}{3}\mathbf{M}] & \text{outside} \end{cases}$$

Griffiths also shows in his Example 4.2 that a uniformly polarized sphere has electric field

$$(0.7) \quad \mathbf{E} = \begin{cases} -\frac{1}{3\epsilon_0}\mathbf{P} & \text{inside} \\ \frac{1}{4\pi\epsilon_0 r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}] & \text{outside} \end{cases}$$

where the total electric dipole moment is

$$(0.8) \quad \mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}$$

so

$$(0.9) \quad \mathbf{E} = \begin{cases} -\frac{1}{3\epsilon_0}\mathbf{P} & \text{inside} \\ \frac{R^3}{\epsilon_0 r^3} [(\mathbf{P} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \frac{1}{3}\mathbf{P}] & \text{outside} \end{cases}$$

To get the momentum density we use 0.1

$$(0.10) \quad \mathbf{p}_{em} = \begin{cases} -\frac{2}{9}\mu_0\mathbf{P} \times \mathbf{M} & \text{inside} \\ \frac{\mu_0 R^6}{r^6} [(\mathbf{P} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \frac{1}{3}\mathbf{P}] \times [(\mathbf{M} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \frac{1}{3}\mathbf{M}] & \text{outside} \end{cases}$$

The momentum density is a constant inside the sphere. The outside density requires a bit of vector juggling to get it into a useful form. We start with the identity

$$(0.11) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Putting $\mathbf{A} = \hat{\mathbf{r}}$, $\mathbf{B} = \mathbf{P}$ and $\mathbf{C} = \mathbf{M}$, we get

$$(0.12) \quad \hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M}) = (\hat{\mathbf{r}} \cdot \mathbf{M})\mathbf{P} - (\hat{\mathbf{r}} \cdot \mathbf{P})\mathbf{M}$$

Taking the cross product again:

$$(0.13) \quad \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M})) = (\hat{\mathbf{r}} \cdot \mathbf{M})(\hat{\mathbf{r}} \times \mathbf{P}) - (\hat{\mathbf{r}} \cdot \mathbf{P})(\hat{\mathbf{r}} \times \mathbf{M})$$

$$(0.14) \quad = -[(\hat{\mathbf{r}} \cdot \mathbf{M})(\mathbf{P} \times \hat{\mathbf{r}}) + (\hat{\mathbf{r}} \cdot \mathbf{P})(\hat{\mathbf{r}} \times \mathbf{M})]$$

Expanding the RHS of 0.10 and using $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$ we get

(0.15)

$$\left[(\mathbf{P} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \frac{1}{3} \mathbf{P} \right] \times \left[(\mathbf{M} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \frac{1}{3} \mathbf{M} \right] = -\frac{1}{3} [(\hat{\mathbf{r}} \cdot \mathbf{M}) (\mathbf{P} \times \hat{\mathbf{r}}) + (\hat{\mathbf{r}} \cdot \mathbf{P}) (\hat{\mathbf{r}} \times \mathbf{M})] + \frac{1}{9} \mathbf{P} \times \mathbf{M}$$

(0.16)

$$= \frac{1}{3} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M})) + \frac{1}{9} \mathbf{P} \times \mathbf{M}$$

So

(0.17)

$$\mathbf{p}_{em} = \begin{cases} -\frac{2}{9} \mu_0 \mathbf{P} \times \mathbf{M} & \text{inside} \\ \frac{\mu_0 R^6}{3r^6} [\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M})) + \frac{1}{3} \mathbf{P} \times \mathbf{M}] & \text{outside} \end{cases}$$

Without loss of generality we can choose the z axis to lie along $\mathbf{P} \times \mathbf{M}$. In that case, the spherical angle θ is the angle between $\hat{\mathbf{r}}$ and $\mathbf{P} \times \mathbf{M}$, so we have

(0.18)

$$\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M}) = -|\mathbf{P} \times \mathbf{M}| \sin \theta \hat{\phi}$$

(0.19)

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M})) = |\mathbf{P} \times \mathbf{M}| \sin \theta \hat{\theta}$$

To get the total momentum in the field, we need to integrate 0.17 over all space, so we get (using the symbol \mathfrak{P} for total momentum)

(0.20)

$$\mathfrak{P} = \begin{cases} -\frac{4}{3} \pi R^3 \frac{2}{9} \mu_0 \mathbf{P} \times \mathbf{M} = -\frac{8}{27} \pi \mu_0 R^3 \mathbf{P} \times \mathbf{M} & \text{inside} \\ \frac{\mu_0 R^6}{3} \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{r^2}{r^6} [|\mathbf{P} \times \mathbf{M}| \sin \theta \hat{\theta} + \frac{1}{3} \mathbf{P} \times \mathbf{M}] \sin \theta d\phi d\theta dr & \text{outside} \end{cases}$$

To do the integral, we need to express $\hat{\theta}$ in rectangular coordinates

(0.21)

$$\hat{\theta} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}$$

The ϕ integral will kill off the x and y components, so we are left with

(0.22)

$$\mathfrak{P}_{out} = \frac{2\pi\mu_0 R^6}{3} \int_R^\infty \frac{dr}{r^4} \int_0^\pi \left[-|\mathbf{P} \times \mathbf{M}| \sin^3 \theta \hat{\mathbf{z}} + \frac{1}{3} \mathbf{P} \times \mathbf{M} \sin \theta \right] d\theta$$

(0.23)

$$= \frac{2\pi\mu_0 R^6}{3} \mathbf{P} \times \mathbf{M} \int_R^\infty \frac{dr}{r^4} \int_0^\pi \left[-\sin^3 \theta + \frac{1}{3} \sin \theta \right] d\theta$$

(0.24)

$$= \frac{2\pi\mu_0 R^3}{9} \mathbf{P} \times \mathbf{M} \left(-\frac{4}{3} + \frac{2}{3} \right)$$

(0.25)

$$= -\frac{4}{27} \pi \mu_0 R^3 \mathbf{P} \times \mathbf{M}$$

The total momentum is thus

$$(0.26) \quad \mathfrak{P} = -\frac{8}{27}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M} - \frac{4}{27}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M}$$

$$(0.27) \quad = -\frac{4}{9}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M}$$

$$(0.28) \quad = \frac{4}{9}\pi\mu_0 R^3 \mathbf{M} \times \mathbf{P}$$