MOMENTUM IN A MAGNETIZED AND POLARIZED SPHERE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.10.

Here's another example of finding momentum in electromagnetic fields. Consider a non-conducting sphere of radius R with a uniform magnetization $\mathbf{M}$ and a uniform polarization $\mathbf{P}$, which need not be in the same direction as $\mathbf{M}$. What is the total momentum stored in the fields?

The momentum density is

$$\mathfrak{p}_{em} = \varepsilon_0 \mathbf{E} \times \mathbf{B} \tag{1}$$

It's easiest to work with the fields in their coordinate free forms. The field of a pure magnetic dipole at the origin is

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{m}\right] \tag{2}$$

and for a pure electric dipole we have

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right] \tag{3}$$

Griffiths shows in his Example 6.1 that the magnetic field of a uniformly magnetized sphere is

$$\mathbf{B} = \begin{cases} \frac{2}{3}\mu_0\mathbf{M} & \text{inside} \\ \frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{m}\right] & \text{outside} \end{cases} \tag{4}$$

where the total magnetic dipole moment of the sphere is

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M} \tag{5}$$

so

$$\mathbf{B}=\begin{cases}\frac{2}{3}\mu_0\mathbf{M} & \text{inside}\\ \frac{\mu_0R^3}{r^3}\left[\left(\mathbf{M}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\frac{1}{3}\mathbf{M}\right] & \text{outside}\end{cases}\tag{6}$$

Griffiths also shows in his Example 4.2 that a uniformly polarized sphere has electric field

$$\mathbf{E}=\begin{cases}-\frac{1}{3\varepsilon_0}\mathbf{P} & \text{inside}\\ \frac{1}{4\pi\varepsilon_0r^3}\left[3\left(\mathbf{p}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{p}\right] & \text{outside}\end{cases}\tag{7}$$

where the total electric dipole moment is

$$\mathbf{p}=\frac{4}{3}\pi R^3\mathbf{P}\tag{8}$$

so

$$\mathbf{E}=\begin{cases}-\frac{1}{3\varepsilon_0}\mathbf{P} & \text{inside}\\ \frac{R^3}{\varepsilon_0r^3}\left[\left(\mathbf{P}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\frac{1}{3}\mathbf{P}\right] & \text{outside}\end{cases}\tag{9}$$

To get the momentum density we use 1

$$\mathfrak{p}_{em}=\begin{cases}-\frac{2}{9}\mu_0\mathbf{P}\times\mathbf{M} & \text{inside}\\ \frac{\mu_0R^6}{r^6}\left[\left(\mathbf{P}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\frac{1}{3}\mathbf{P}\right]\times\left[\left(\mathbf{M}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\frac{1}{3}\mathbf{M}\right] & \text{outside}\end{cases}\tag{10}$$

The momentum density is a constant inside the sphere. The outside density requires a bit of vector juggling to get it into a useful form. We start with the identity

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\left(\mathbf{A}\cdot\mathbf{C}\right)\mathbf{B}-\left(\mathbf{A}\cdot\mathbf{B}\right)\mathbf{C}\tag{11}$$

Putting $\mathbf{A}=\hat{\mathbf{r}}$, $\mathbf{B}=\mathbf{P}$ and $\mathbf{C}=\mathbf{M}$, we get

$$\hat{\mathbf{r}}\times\left(\mathbf{P}\times\mathbf{M}\right)=\left(\hat{\mathbf{r}}\cdot\mathbf{M}\right)\mathbf{P}-\left(\hat{\mathbf{r}}\cdot\mathbf{P}\right)\mathbf{M}\tag{12}$$

Taking the cross product again:

$$\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\left(\mathbf{P}\times\mathbf{M}\right)\right)=\left(\hat{\mathbf{r}}\cdot\mathbf{M}\right)\left(\hat{\mathbf{r}}\times\mathbf{P}\right)-\left(\hat{\mathbf{r}}\cdot\mathbf{P}\right)\left(\hat{\mathbf{r}}\times\mathbf{M}\right)\tag{13}$$

$$=-\left[\left(\hat{\mathbf{r}}\cdot\mathbf{M}\right)\left(\mathbf{P}\times\hat{\mathbf{r}}\right)+\left(\hat{\mathbf{r}}\cdot\mathbf{P}\right)\left(\hat{\mathbf{r}}\times\mathbf{M}\right)\right]\tag{14}$$

Expanding the RHS of 10 and using $\hat{\mathbf{r}}\times\hat{\mathbf{r}}=0$ we get

$$\left[(\mathbf{P}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\frac{1}{3}\mathbf{P}\right]\times\left[(\mathbf{M}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\frac{1}{3}\mathbf{M}\right]=-\frac{1}{3}\left[(\hat{\mathbf{r}}\cdot\mathbf{M})(\mathbf{P}\times\hat{\mathbf{r}})+(\hat{\mathbf{r}}\cdot\mathbf{P})(\hat{\mathbf{r}}\times\mathbf{M})\right]+\frac{1}{9}\mathbf{P}\times\mathbf{M} \tag{15}$$

$$=\frac{1}{3}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times(\mathbf{P}\times\mathbf{M}))+\frac{1}{9}\mathbf{P}\times\mathbf{M} \tag{16}$$

So

$$\mathfrak{p}_{em}=\begin{cases}-\frac{2}{9}\mu_0\mathbf{P}\times\mathbf{M} & \text{inside}\\ \frac{\mu_0R^6}{3r^6}\left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times(\mathbf{P}\times\mathbf{M}))+\frac{1}{3}\mathbf{P}\times\mathbf{M}\right] & \text{outside}\end{cases} \tag{17}$$

Without loss of generality we can choose the z axis to lie along $\mathbf{P}\times\mathbf{M}$. In that case, the spherical angle θ is the angle between $\hat{\mathbf{r}}$ and $\mathbf{P}\times\mathbf{M}$, so we have

$$\hat{\mathbf{r}}\times(\mathbf{P}\times\mathbf{M}) = -|\mathbf{P}\times\mathbf{M}|\sin\theta\hat{\phi} \tag{18}$$

$$\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times(\mathbf{P}\times\mathbf{M})) = |\mathbf{P}\times\mathbf{M}|\sin\theta\hat{\theta} \tag{19}$$

To get the total momentum in the field, we need to integrate 17 over all space, so we get (using the symbol $\mathfrak{P}$ for total momentum)

$$\mathfrak{P}=\begin{cases}-\frac{4}{3}\pi R^3\frac{2}{9}\mu_0\mathbf{P}\times\mathbf{M}=-\frac{8}{27}\pi\mu_0R^3\mathbf{P}\times\mathbf{M} & \text{inside}\\ \frac{\mu_0R^6}{3}\int_R^\infty\int_0^\pi\int_0^{2\pi}\frac{r^2}{r^6}\left[|\mathbf{P}\times\mathbf{M}|\sin\theta\hat{\theta}+\frac{1}{3}\mathbf{P}\times\mathbf{M}\right]\sin\theta d\phi d\theta dr & \text{outside}\end{cases} \tag{20}$$

To do the integral, we need to express $\hat{\theta}$ in rectangular coordinates

$$\hat{\theta}=\cos\theta\cos\phi\hat{\mathbf{x}}+\cos\theta\sin\phi\hat{\mathbf{y}}-\sin\theta\hat{\mathbf{z}} \tag{21}$$

The ϕ integral will kill off the x and y components, so we are left with

$$\mathfrak{P}_{out}=\frac{2\pi\mu_0R^6}{3}\int_R^\infty\frac{dr}{r^4}\int_0^\pi\left[-|\mathbf{P}\times\mathbf{M}|\sin^3\theta\hat{\mathbf{z}}+\frac{1}{3}\mathbf{P}\times\mathbf{M}\sin\theta\right]d\theta \tag{22}$$

$$=\frac{2\pi\mu_0R^6}{3}\mathbf{P}\times\mathbf{M}\int_R^\infty\frac{dr}{r^4}\int_0^\pi\left[-\sin^3\theta+\frac{1}{3}\sin\theta\right]d\theta \tag{23}$$

$$=\frac{2\pi\mu_0R^3}{9}\mathbf{P}\times\mathbf{M}\left(-\frac{4}{3}+\frac{2}{3}\right) \tag{24}$$

$$=-\frac{4}{27}\pi\mu_0R^3\mathbf{P}\times\mathbf{M} \tag{25}$$

The total momentum is thus

$$\mathfrak{P} = -\frac{8}{27}\pi\mu_0 R^3 \mathbf{P}\times\mathbf{M} - \frac{4}{27}\pi\mu_0 R^3 \mathbf{P}\times\mathbf{M} \tag{26}$$

$$= -\frac{4}{9}\pi\mu_0 R^3 \mathbf{P}\times\mathbf{M} \tag{27}$$

$$= \frac{4}{9}\pi\mu_0 R^3 \mathbf{M}\times\mathbf{P} \tag{28}$$