

MOMENTUM IN A MAGNETIZED AND POLARIZED SPHERE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.10.

Here's another example of finding momentum in electromagnetic fields. Consider a non-conducting sphere of radius R with a uniform magnetization \mathbf{M} and a uniform polarization \mathbf{P} , which need not be in the same direction as \mathbf{M} . What is the total momentum stored in the fields?

The momentum density is

$$\mathbf{p}_{em} = \epsilon_0 \mathbf{E} \times \mathbf{B} \quad (1)$$

It's easiest to work with the fields in their coordinate free forms. The field of a pure magnetic dipole at the origin is

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}] \quad (2)$$

and for a pure electric dipole we have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}] \quad (3)$$

Griffiths shows in his Example 6.1 that the magnetic field of a uniformly magnetized sphere is

$$\mathbf{B} = \begin{cases} \frac{2}{3}\mu_0\mathbf{M} & \text{inside} \\ \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}] & \text{outside} \end{cases} \quad (4)$$

where the total magnetic dipole moment of the sphere is

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M} \quad (5)$$

so

$$\mathbf{B} = \begin{cases} \frac{2}{3}\mu_0\mathbf{M} & \text{inside} \\ \frac{\mu_0 R^3}{r^3} [(\mathbf{M} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \frac{1}{3}\mathbf{M}] & \text{outside} \end{cases} \quad (6)$$

Griffiths also shows in his Example 4.2 that a uniformly polarized sphere has electric field

$$\mathbf{E} = \begin{cases} -\frac{1}{3\epsilon_0}\mathbf{P} & \text{inside} \\ \frac{1}{4\pi\epsilon_0 r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] & \text{outside} \end{cases} \quad (7)$$

where the total electric dipole moment is

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P} \quad (8)$$

so

$$\mathbf{E} = \begin{cases} -\frac{1}{3\epsilon_0}\mathbf{P} & \text{inside} \\ \frac{R^3}{\epsilon_0 r^3} [(\mathbf{P} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \frac{1}{3}\mathbf{P}] & \text{outside} \end{cases} \quad (9)$$

To get the momentum density we use 1

$$\mathbf{p}_{em} = \begin{cases} -\frac{2}{9}\mu_0 \mathbf{P} \times \mathbf{M} & \text{inside} \\ \frac{\mu_0 R^6}{r^6} [(\mathbf{P} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \frac{1}{3}\mathbf{P}] \times [(\mathbf{M} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \frac{1}{3}\mathbf{M}] & \text{outside} \end{cases} \quad (10)$$

The momentum density is a constant inside the sphere. The outside density requires a bit of vector juggling to get it into a useful form. We start with the identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \quad (11)$$

Putting $\mathbf{A} = \hat{\mathbf{r}}$, $\mathbf{B} = \mathbf{P}$ and $\mathbf{C} = \mathbf{M}$, we get

$$\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M}) = (\hat{\mathbf{r}} \cdot \mathbf{M})\mathbf{P} - (\hat{\mathbf{r}} \cdot \mathbf{P})\mathbf{M} \quad (12)$$

Taking the cross product again:

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M})) = (\hat{\mathbf{r}} \cdot \mathbf{M})(\hat{\mathbf{r}} \times \mathbf{P}) - (\hat{\mathbf{r}} \cdot \mathbf{P})(\hat{\mathbf{r}} \times \mathbf{M}) \quad (13)$$

$$= -[(\hat{\mathbf{r}} \cdot \mathbf{M})(\mathbf{P} \times \hat{\mathbf{r}}) + (\hat{\mathbf{r}} \cdot \mathbf{P})(\hat{\mathbf{r}} \times \mathbf{M})] \quad (14)$$

Expanding the RHS of 10 and using $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$ we get

$$\left[(\mathbf{P} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \frac{1}{3}\mathbf{P} \right] \times \left[(\mathbf{M} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \frac{1}{3}\mathbf{M} \right] = -\frac{1}{3} [(\hat{\mathbf{r}} \cdot \mathbf{M})(\mathbf{P} \times \hat{\mathbf{r}}) + (\hat{\mathbf{r}} \cdot \mathbf{P})(\hat{\mathbf{r}} \times \mathbf{M})] + \frac{1}{9}\mathbf{P} \times \mathbf{M} \quad (15)$$

$$= \frac{1}{3}\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M})) + \frac{1}{9}\mathbf{P} \times \mathbf{M} \quad (16)$$

So

$$\mathbf{p}_{em} = \begin{cases} -\frac{2}{9}\mu_0\mathbf{P} \times \mathbf{M} & \text{inside} \\ \frac{\mu_0 R^6}{3r^6} [\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M}))] + \frac{1}{3}\mathbf{P} \times \mathbf{M} & \text{outside} \end{cases} \quad (17)$$

Without loss of generality we can choose the z axis to lie along $\mathbf{P} \times \mathbf{M}$. In that case, the spherical angle θ is the angle between $\hat{\mathbf{r}}$ and $\mathbf{P} \times \mathbf{M}$, so we have

$$\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M}) = -|\mathbf{P} \times \mathbf{M}| \sin \theta \hat{\phi} \quad (18)$$

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\mathbf{P} \times \mathbf{M})) = |\mathbf{P} \times \mathbf{M}| \sin \theta \hat{\theta} \quad (19)$$

To get the total momentum in the field, we need to integrate 17 over all space, so we get (using the symbol \mathfrak{P} for total momentum)

$$\mathfrak{P} = \begin{cases} -\frac{4}{3}\pi R^3 \frac{2}{9}\mu_0\mathbf{P} \times \mathbf{M} = -\frac{8}{27}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M} & \text{inside} \\ \frac{\mu_0 R^6}{3} \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{r^2}{r^6} [|\mathbf{P} \times \mathbf{M}| \sin \theta \hat{\theta} + \frac{1}{3}\mathbf{P} \times \mathbf{M}] \sin \theta d\phi d\theta dr & \text{outside} \end{cases} \quad (20)$$

To do the integral, we need to express $\hat{\theta}$ in rectangular coordinates

$$\hat{\theta} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \quad (21)$$

The ϕ integral will kill off the x and y components, so we are left with

$$\mathfrak{P}_{out} = \frac{2\pi\mu_0 R^6}{3} \int_R^\infty \frac{dr}{r^4} \int_0^\pi \left[-|\mathbf{P} \times \mathbf{M}| \sin^3 \theta \hat{\mathbf{z}} + \frac{1}{3}\mathbf{P} \times \mathbf{M} \sin \theta \right] d\theta \quad (22)$$

$$= \frac{2\pi\mu_0 R^6}{3} \mathbf{P} \times \mathbf{M} \int_R^\infty \frac{dr}{r^4} \int_0^\pi \left[-\sin^3 \theta + \frac{1}{3} \sin \theta \right] d\theta \quad (23)$$

$$= \frac{2\pi\mu_0 R^3}{9} \mathbf{P} \times \mathbf{M} \left(-\frac{4}{3} + \frac{2}{3} \right) \quad (24)$$

$$= -\frac{4}{27}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M} \quad (25)$$

The total momentum is thus

$$\mathfrak{P} = -\frac{8}{27}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M} - \frac{4}{27}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M} \quad (26)$$

$$= -\frac{4}{9}\pi\mu_0 R^3 \mathbf{P} \times \mathbf{M} \quad (27)$$

$$= \frac{4}{9}\pi\mu_0 R^3 \mathbf{M} \times \mathbf{P} \quad (28)$$