THE ELECTRON AS ELECTROMAGNETIC ENERGY AND ANGULAR MOMENTUM

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.11.

Early in the twentieth century, a classical model of the electron was proposed in which the electron was taken to be a spherical shell of charge of radius R spinning with angular speed ω, and that its entire mass (strictly, mass-energy) was composed of the energy stored in its electromagnetic fields, and its angular momentum was that of the electromagentic fields as well.

First, we work out the energy, which is

$$W=\frac{1}{2}\int_{\mathcal{V}}\left(\frac{1}{\mu_0}B^2+\varepsilon_0E^2\right)d^3\mathbf{r} \tag{1}$$

The electric field of a spherical shell is

$$\mathbf{E}=\begin{cases}0 & r<R\\ \frac{q}{4\pi\varepsilon_0r^2}\hat{\mathbf{r}} & r>R\end{cases} \tag{2}$$

Griffiths works out the magnetic vector potential of a spinning spherical shell of charge in his Example 5.11:

$$\mathbf{A}=\begin{cases}\frac{1}{3}\mu_0R\omega\sigma r\sin\theta\hat{\phi} & r\le R\\ \frac{1}{3}\mu_0R^4\omega\sigma\frac{\sin\theta}{r^2}\hat{\phi} & r\ge R\end{cases} \tag{3}$$

from which we can get $\mathbf{B}$ from

$$\mathbf{B}=\nabla\times\mathbf{A}=\begin{cases}\frac{2}{3}\mu_0R\omega\sigma\hat{\mathbf{z}} & r\le R\\ \frac{1}{3}\frac{\mu_0R^4\omega\sigma}{r^3}\left[2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right] & r\ge R\end{cases} \tag{4}$$

The surface charge density $\sigma=q/4\pi R^2$ so we get

$$\mathbf{B}=\begin{cases}\frac{\mu_0q\omega}{6\pi R}\hat{\mathbf{z}} & r\le R\\ \frac{\mu_0qR^2\omega}{12\pi r^3}\left[2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right] & r\ge R\end{cases} \tag{5}$$

The energy is

$$W = \frac{1}{2\mu_0}\frac{4}{3}\pi R^3\left(\frac{\mu_0 q\omega}{6\pi R}\right)^2 + \frac{1}{2\mu_0}\left(\frac{\mu_0 q R^2\omega}{12\pi}\right)^2(2\pi)\int_R^\infty\int_0^\pi \frac{4\cos^2\theta+\sin^2\theta}{r^6}r^2\sin\theta d\theta dr+ \tag{6}$$

$$\frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2(2\pi)\int_R^\infty\int_0^\pi \frac{1}{r^6}r^2\sin\theta d\theta dr$$

$$= \frac{\mu_0}{54\pi}q^2R\omega^2 + \frac{\mu_0 q^2R^4\omega^2}{144\pi}\int_R^\infty\int_0^\pi \frac{3\cos^2\theta+1}{r^4}\sin\theta d\theta dr + \frac{q^2}{16\pi\varepsilon_0}\int_R^\infty\int_0^\pi \frac{\sin\theta}{r^4}d\theta dr \tag{7}$$

$$= \frac{\mu_0 q^2R\omega^2}{54\pi} + \frac{\mu_0 q^2R\omega^2}{108\pi} + \frac{q^2}{8\pi\varepsilon_0 R} \tag{8}$$

$$= \frac{3\mu_0 q^2R\omega^2}{108\pi} + \frac{q^2}{8\pi\varepsilon_0 R} \tag{9}$$

To calculate the angular momentum, we first need the linear momentum. The momentum density is

$$\mathfrak{p}_{em} = \varepsilon_0\mathbf{E}\times\mathbf{B} \tag{10}$$

Only the region outside the sphere contributes (since $\mathbf{E} = 0$ inside), so we get

$$\mathfrak{p}_{em} = \frac{\mu_0 q^2R^2\omega}{48\pi^2}\frac{\sin\theta}{r^5}\hat{\phi} \tag{11}$$

$$\mathfrak{L}_{em} = \mathbf{r}\times\mathfrak{p}_{em} \tag{12}$$

$$= -\frac{\mu_0 q^2R^2\omega}{48\pi^2}\frac{\sin\theta}{r^4}\hat{\theta} \tag{13}$$

The total angular momentum is

$$\mathbf{L}_{em} = -\frac{\mu_0 q^2R^2\omega}{48\pi^2}\int_R^\infty\int_0^\pi\int_0^{2\pi}\frac{\sin\theta}{r^4}\hat{\theta}r^2\sin\theta d\phi d\theta dr \tag{14}$$

To do the integral, we need to express $\hat{\theta}$ in rectangular coordinates

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \tag{15}$$

The ϕ integral will kill off the x and y components, so we are left with

$$\mathbf{L}_{em} = \frac{\mu_0 q^2 R^2 \omega}{24\pi}\hat{\mathbf{z}} \int_R^\infty \int_0^\pi \frac{\sin^3\theta}{r^2} d\theta dr \tag{16}$$
$$= \frac{\mu_0 q^2 R\omega}{18\pi}\hat{\mathbf{z}} \tag{17}$$

The idea now is to equate the energy with relativistic rest mass and the angular momentum with the quantum mechanical spin, so we get from 9 and 17

$$\frac{3\mu_0 q^2 R\omega^2}{108\pi} + \frac{q^2}{8\pi\varepsilon_0 R} = mc^2 \tag{18}$$
$$\frac{\mu_0 q^2 R\omega}{18\pi} = \frac{\hbar}{2} \tag{19}$$

Solving for R and ω we get

$$R = \frac{1}{8}\frac{18\hbar^2\pi^2\varepsilon_0 + q^4\mu_0}{mc^2\pi\,\varepsilon_0\mu_0 q^2} \tag{20}$$
$$\omega = 72\,\frac{\hbar\,\pi^2 mc^2\varepsilon_0}{18\hbar^2\pi^2\varepsilon_0 + q^4\mu_0} \tag{21}$$

We can plug in the values of the various constants (all in SI units):

$$\hbar = 1.0546\times 10^{-34} \tag{22}$$
$$m = 9.109\times 10^{-31} \tag{23}$$
$$\varepsilon_0 = 8.854\times 10^{-12} \tag{24}$$
$$\mu_0 = 1.2566\times 10^{-6} \tag{25}$$
$$c = 3.0\times 10^{8} \tag{26}$$

and we get

$$R = 2.978\times 10^{-11}\text{ m} \tag{27}$$
$$\omega = 3.105\times 10^{21}\text{ s}^{-1} \tag{28}$$

This gives an equatorial speed of

$$\omega R = 9.246\times 10^{10}\text{ m s}^{-1} \tag{29}$$

which is more than 300 times the speed of light, so this model of the electron doesn't work.