

THE ELECTRON AS ELECTROMAGNETIC ENERGY AND ANGULAR MOMENTUM

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.11.

Early in the twentieth century, a classical model of the electron was proposed in which the electron was taken to be a spherical shell of charge of radius R spinning with angular speed ω , and that its entire mass (strictly, mass-energy) was composed of the energy stored in its electromagnetic fields, and its angular momentum was that of the electromagnetic fields as well.

First, we work out the energy, which is

$$W = \frac{1}{2} \int_V \left(\frac{1}{\mu_0} B^2 + \epsilon_0 E^2 \right) d^3 \mathbf{r} \quad (1)$$

The electric field of a spherical shell is

$$\mathbf{E} = \begin{cases} 0 & r < R \\ \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} & r > R \end{cases} \quad (2)$$

Griffiths works out the magnetic vector potential of a spinning spherical shell of charge in his Example 5.11:

$$\mathbf{A} = \begin{cases} \frac{1}{3} \mu_0 R \omega \sigma r \sin \theta \hat{\phi} & r \leq R \\ \frac{1}{3} \mu_0 R^4 \omega \sigma \frac{\sin \theta}{r^2} \hat{\phi} & r \geq R \end{cases} \quad (3)$$

from which we can get \mathbf{B} from

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{cases} \frac{2}{3} \mu_0 R \omega \sigma \hat{\mathbf{z}} & r \leq R \\ \frac{1}{3} \frac{\mu_0 R^4 \omega \sigma}{r^3} [2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}] & r \geq R \end{cases} \quad (4)$$

The surface charge density $\sigma = q/4\pi R^2$ so we get

$$\mathbf{B} = \begin{cases} \frac{\mu_0 q \omega}{6\pi R} \hat{\mathbf{z}} & r \leq R \\ \frac{\mu_0 q R^2 \omega}{12\pi r^3} [2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}] & r \geq R \end{cases} \quad (5)$$

The energy is

$$W = \frac{1}{2\mu_0} \frac{4}{3} \pi R^3 \left(\frac{\mu_0 q \omega}{6\pi R} \right)^2 + \frac{1}{2\mu_0} \left(\frac{\mu_0 q R^2 \omega}{12\pi} \right)^2 (2\pi) \int_R^\infty \int_0^\pi \frac{4\cos^2\theta + \sin^2\theta}{r^6} r^2 \sin\theta d\theta dr + \quad (6)$$

$$\begin{aligned} & \frac{\epsilon_0}{2} \left(\frac{q}{4\pi\epsilon_0} \right)^2 (2\pi) \int_R^\infty \int_0^\pi \frac{1}{r^6} r^2 \sin\theta d\theta dr \\ &= \frac{\mu_0}{54\pi} q^2 R \omega^2 + \frac{\mu_0 q^2 R^4 \omega^2}{144\pi} \int_R^\infty \int_0^\pi \frac{3\cos^2\theta + 1}{r^4} \sin\theta d\theta dr + \frac{q^2}{16\pi\epsilon_0} \int_R^\infty \int_0^\pi \frac{\sin\theta}{r^4} d\theta dr \end{aligned} \quad (7)$$

$$= \frac{\mu_0 q^2 R \omega^2}{54\pi} + \frac{\mu_0 q^2 R \omega^2}{108\pi} + \frac{q^2}{8\pi\epsilon_0 R} \quad (8)$$

$$= \frac{3\mu_0 q^2 R \omega^2}{108\pi} + \frac{q^2}{8\pi\epsilon_0 R} \quad (9)$$

To calculate the angular momentum, we first need the linear momentum. The momentum density is

$$\mathbf{p}_{em} = \epsilon_0 \mathbf{E} \times \mathbf{B} \quad (10)$$

Only the region outside the sphere contributes (since $\mathbf{E} = 0$ inside), so we get

$$\mathbf{p}_{em} = \frac{\mu_0 q^2 R^2 \omega \sin\theta}{48\pi^2} \frac{\hat{\phi}}{r^5} \quad (11)$$

$$\mathfrak{L}_{em} = \mathbf{r} \times \mathbf{p}_{em} \quad (12)$$

$$= -\frac{\mu_0 q^2 R^2 \omega \sin\theta}{48\pi^2} \frac{\hat{\theta}}{r^4} \quad (13)$$

The total angular momentum is

$$\mathbf{L}_{em} = -\frac{\mu_0 q^2 R^2 \omega}{48\pi^2} \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{\sin\theta}{r^4} \hat{\theta} r^2 \sin\theta d\phi d\theta dr \quad (14)$$

To do the integral, we need to express $\hat{\theta}$ in rectangular coordinates

$$\hat{\theta} = \cos\theta \cos\phi \hat{\mathbf{x}} + \cos\theta \sin\phi \hat{\mathbf{y}} - \sin\theta \hat{\mathbf{z}} \quad (15)$$

The ϕ integral will kill off the x and y components, so we are left with

$$\mathbf{L}_{em} = \frac{\mu_0 q^2 R^2 \omega}{24\pi} \hat{\mathbf{z}} \int_R^\infty \int_0^\pi \frac{\sin^3 \theta}{r^2} d\theta dr \quad (16)$$

$$= \frac{\mu_0 q^2 R \omega}{18\pi} \hat{\mathbf{z}} \quad (17)$$

The idea now is to equate the energy with relativistic rest mass and the angular momentum with the quantum mechanical spin, so we get from 9 and 17

$$\frac{3\mu_0 q^2 R \omega^2}{108\pi} + \frac{q^2}{8\pi\epsilon_0 R} = mc^2 \quad (18)$$

$$\frac{\mu_0 q^2 R \omega}{18\pi} = \frac{\hbar}{2} \quad (19)$$

Solving for R and ω we get

$$R = \frac{1}{8} \frac{18 \hbar^2 \pi^2 \epsilon_0 + q^4 \mu_0}{mc^2 \pi \epsilon_0 \mu_0 q^2} \quad (20)$$

$$\omega = 72 \frac{\hbar \pi^2 mc^2 \epsilon_0}{18 \hbar^2 \pi^2 \epsilon_0 + q^4 \mu_0} \quad (21)$$

We can plug in the values of the various constants (all in SI units):

$$\hbar = 1.0546 \times 10^{-34} \quad (22)$$

$$m = 9.109 \times 10^{-31} \quad (23)$$

$$\epsilon_0 = 8.854 \times 10^{-12} \quad (24)$$

$$\mu_0 = 1.2566 \times 10^{-6} \quad (25)$$

$$c = 3.0 \times 10^8 \quad (26)$$

and we get

$$R = 2.978 \times 10^{-11} \text{ m} \quad (27)$$

$$\omega = 3.105 \times 10^{21} \text{ s}^{-1} \quad (28)$$

This gives an equatorial speed of

$$\omega R = 9.246 \times 10^{10} \text{ m s}^{-1} \quad (29)$$

which is more than 300 times the speed of light, so this model of the electron doesn't work.