

THE ELECTRON AS ELECTROMAGNETIC ENERGY AND ANGULAR MOMENTUM

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.11.

Early in the twentieth century, a classical model of the electron was proposed in which the electron was taken to be a spherical shell of charge of radius R spinning with angular speed ω , and that its entire mass (strictly, mass-energy) was composed of the energy stored in its electromagnetic fields, and its angular momentum was that of the electromagnetic fields as well.

First, we work out the energy, which is

$$(1) \quad W = \frac{1}{2} \int_V \left(\frac{1}{\mu_0} B^2 + \epsilon_0 E^2 \right) d^3 \mathbf{r}$$

The electric field of a spherical shell is

$$(2) \quad \mathbf{E} = \begin{cases} 0 & r < R \\ \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} & r > R \end{cases}$$

Griffiths works out the magnetic vector potential of a spinning spherical shell of charge in his Example 5.11:

$$(3) \quad \mathbf{A} = \begin{cases} \frac{1}{3} \mu_0 R \omega \sigma r \sin \theta \hat{\phi} & r \leq R \\ \frac{1}{3} \mu_0 R^4 \omega \sigma \frac{\sin \theta}{r^2} \hat{\phi} & r \geq R \end{cases}$$

from which we can get \mathbf{B} from

$$(4) \quad \mathbf{B} = \nabla \times \mathbf{A} = \begin{cases} \frac{2}{3} \mu_0 R \omega \sigma \hat{\mathbf{z}} & r \leq R \\ \frac{1}{3} \frac{\mu_0 R^4 \omega \sigma}{r^3} [2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}] & r \geq R \end{cases}$$

The surface charge density $\sigma = q/4\pi R^2$ so we get

$$(5) \quad \mathbf{B} = \begin{cases} \frac{\mu_0 q \omega}{6\pi R} \hat{\mathbf{z}} & r \leq R \\ \frac{\mu_0 q R^2 \omega}{12\pi r^3} [2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}] & r \geq R \end{cases}$$

The energy is

(6)

$$W = \frac{1}{2\mu_0} \frac{4}{3} \pi R^3 \left(\frac{\mu_0 q \omega}{6\pi R} \right)^2 + \frac{1}{2\mu_0} \left(\frac{\mu_0 q R^2 \omega}{12\pi} \right)^2 (2\pi) \int_R^\infty \int_0^\pi \frac{4\cos^2\theta + \sin^2\theta}{r^6} r^2 \sin\theta d\theta dr +$$

$$\frac{\epsilon_0}{2} \left(\frac{q}{4\pi\epsilon_0} \right)^2 (2\pi) \int_R^\infty \int_0^\pi \frac{1}{r^6} r^2 \sin\theta d\theta dr$$

(7)

$$= \frac{\mu_0}{54\pi} q^2 R \omega^2 + \frac{\mu_0 q^2 R^4 \omega^2}{144\pi} \int_R^\infty \int_0^\pi \frac{3\cos^2\theta + 1}{r^4} \sin\theta d\theta dr + \frac{q^2}{16\pi\epsilon_0} \int_R^\infty \int_0^\pi \frac{\sin\theta}{r^4} d\theta dr$$

(8)

$$= \frac{\mu_0 q^2 R \omega^2}{54\pi} + \frac{\mu_0 q^2 R \omega^2}{108\pi} + \frac{q^2}{8\pi\epsilon_0 R}$$

(9)

$$= \frac{3\mu_0 q^2 R \omega^2}{108\pi} + \frac{q^2}{8\pi\epsilon_0 R}$$

To calculate the angular momentum, we first need the linear momentum. The momentum density is

(10)

$$\mathbf{p}_{em} = \epsilon_0 \mathbf{E} \times \mathbf{B}$$

Only the region outside the sphere contributes (since $\mathbf{E} = 0$ inside), so we get

(11)

$$\mathbf{p}_{em} = \frac{\mu_0 q^2 R^2 \omega \sin\theta}{48\pi^2} \frac{1}{r^5} \hat{\phi}$$

(12)

$$\mathfrak{L}_{em} = \mathbf{r} \times \mathbf{p}_{em}$$

(13)

$$= -\frac{\mu_0 q^2 R^2 \omega \sin\theta}{48\pi^2} \frac{1}{r^4} \hat{\theta}$$

The total angular momentum is

$$(14) \quad \mathbf{L}_{em} = -\frac{\mu_0 q^2 R^2 \omega}{48\pi^2} \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{\sin\theta}{r^4} \hat{\theta} r^2 \sin\theta d\phi d\theta dr$$

To do the integral, we need to express $\hat{\theta}$ in rectangular coordinates

(15)

$$\hat{\theta} = \cos\theta \cos\phi \hat{x} + \cos\theta \sin\phi \hat{y} - \sin\theta \hat{z}$$

The ϕ integral will kill off the x and y components, so we are left with

$$(16) \quad \mathbf{L}_{em} = \frac{\mu_0 q^2 R^2 \omega}{24\pi} \hat{\mathbf{z}} \int_R^\infty \int_0^\pi \frac{\sin^3 \theta}{r^2} d\theta dr$$

$$(17) \quad = \frac{\mu_0 q^2 R \omega}{18\pi} \hat{\mathbf{z}}$$

The idea now is to equate the energy with relativistic rest mass and the angular momentum with the quantum mechanical spin, so we get from 9 and 17

$$(18) \quad \frac{3\mu_0 q^2 R \omega^2}{108\pi} + \frac{q^2}{8\pi\epsilon_0 R} = mc^2$$

$$(19) \quad \frac{\mu_0 q^2 R \omega}{18\pi} = \frac{\hbar}{2}$$

Solving for R and ω we get

$$(20) \quad R = \frac{1}{8} \frac{18\hbar^2 \pi^2 \epsilon_0 + q^4 \mu_0}{mc^2 \pi \epsilon_0 \mu_0 q^2}$$

$$(21) \quad \omega = 72 \frac{\hbar \pi^2 mc^2 \epsilon_0}{18\hbar^2 \pi^2 \epsilon_0 + q^4 \mu_0}$$

We can plug in the values of the various constants (all in SI units):

$$(22) \quad \hbar = 1.0546 \times 10^{-34}$$

$$(23) \quad m = 9.109 \times 10^{-31}$$

$$(24) \quad \epsilon_0 = 8.854 \times 10^{-12}$$

$$(25) \quad \mu_0 = 1.2566 \times 10^{-6}$$

$$(26) \quad c = 3.0 \times 10^8$$

and we get

$$(27) \quad R = 2.978 \times 10^{-11} \text{ m}$$

$$(28) \quad \omega = 3.105 \times 10^{21} \text{ s}^{-1}$$

This gives an equatorial speed of

$$(29) \quad \omega R = 9.246 \times 10^{10} \text{ m s}^{-1}$$

which is more than 300 times the speed of light, so this model of the electron doesn't work.