ANGULAR MOMENTUM OF A CHARGE AND MAGNETIC MONOPOLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.12.

Returning to the fantasy land where magnetic monopoles exist, suppose we have a single electric charge e and a single magnetic monopole (magnetic charge) b. Then the fields due to these charges are

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{1}$$

$$\mathbf{B} = \frac{\mu_0 b}{4\pi r_b^2}\hat{\mathbf{r}}_b \tag{2}$$

where we've placed e at the origin. The magnetic charge b is not at the origin, so $\hat{\mathbf{r}}_b$ is the vector from b's location to the point where we're measuring the fields. We'd like to find the total angular momentum contained in these two fields.

We'll place b on the z axis a distance d from e, so its location is $d\hat{\mathbf{z}}$. Then the vectors $d\hat{\mathbf{z}}$, $\mathbf{r}$ and $\mathbf{r}_b$ form a triangle with the spherical angle θ lying between $d\hat{\mathbf{z}}$ and $\mathbf{r}$, so it is the angle opposite the side $\mathbf{r}_b$. Therefore

$$\mathbf{r}_b = \mathbf{r} - d\hat{\mathbf{z}} \tag{3}$$

$$= r\hat{\mathbf{r}} - d\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right) \tag{4}$$

$$= (r - d\cos\theta)\,\hat{\mathbf{r}} + d\sin\theta\,\hat{\boldsymbol{\theta}} \tag{5}$$

By the cosine rule for triangles

$$r_b^2 = d^2 + r^2 - 2dr\cos\theta \tag{6}$$

so we get

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{7}$$

$$\mathbf{B} = \frac{\mu_0 b}{4\pi\left(d^2 + r^2 - 2dr\cos\theta\right)^{3/2}}\left[(r - d\cos\theta)\,\hat{\mathbf{r}} + d\sin\theta\,\hat{\boldsymbol{\theta}}\right] \tag{8}$$

The momentum density is

$$\begin{aligned}\mathfrak{p}_{em} &= \epsilon_0 \mathbf{E}\times\mathbf{B} &(9)\\ &= \frac{\mu_0 ebd}{16\pi^2}\frac{\sin\theta}{r^2\left(d^2+r^2-2dr\cos\theta\right)^{3/2}}\hat{\phi} &(10)\end{aligned}$$

and the angular momentum density is

$$\begin{aligned}\mathfrak{L}_{em} &= \mathbf{r}\times\mathfrak{p}_{em} &(11)\\ &= -\frac{\mu_0 ebd}{16\pi^2}\frac{\sin\theta}{r\left(d^2+r^2-2dr\cos\theta\right)^{3/2}}\hat{\theta} &(12)\end{aligned}$$

To find the total angular momentum, we integrate over all space

$$\mathbf{L} = -\frac{\mu_0 ebd}{16\pi^2}\int_0^{\pi}\int_0^{\infty}\int_0^{2\pi}\frac{\sin\theta}{r\left(d^2+r^2-2dr\cos\theta\right)^{3/2}}\hat{\theta}r^2\sin\theta d\phi dr d\theta \tag{13}$$

As usual, we must first express $\hat{\theta}$ in rectangular coordinates

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}}+\cos\theta\sin\phi\hat{\mathbf{y}}-\sin\theta\hat{\mathbf{z}} \tag{14}$$

The ϕ integral will kill off the x and y components, so we are left with

$$\mathbf{L} = \frac{\mu_0 ebd}{16\pi^2}\hat{\mathbf{z}}(2\pi)\int_0^{\pi}\int_0^{\infty}\frac{r\sin^3\theta}{\left(d^2+r^2-2dr\cos\theta\right)^{3/2}}dr d\theta \tag{15}$$

This is a rather unpleasant integral that can be done using Maple, with the result

$$\int_0^{\pi}\int_0^{\infty}\frac{r\sin^3\theta}{\left(d^2+r^2-2dr\cos\theta\right)^{3/2}}dr d\theta = \frac{2}{d} \tag{16}$$

so the angular momentum becomes

$$\mathbf{L} = \frac{\mu_0 eb}{4\pi}\hat{\mathbf{z}} \tag{17}$$

I could leave it at that, but for once I decided to solve the integral by hand, just to see how it's done (since the difficulty of the integral would seem to be the main reason Griffiths has this marked as a 'hard' problem). It seems easiest to do the r integral first, so we can rewrite the integral as

$$\int_0^{\pi}d\theta\sin^3\theta\int_0^{\infty}\frac{r}{\left(d^2+r^2-2dr\cos\theta\right)^{3/2}}dr \tag{18}$$

If we had the derivative of the expression in parentheses in the denominator present in the numerator, the integral would be easy. So we can try something like this (I'll call $\cos\theta \equiv c$ and $\sin\theta \equiv s$ to save a bit of writing; θ is a constant in the r integral anyway so it shouldn't cause any confusion):

$$\int_0^\infty \frac{r}{\left(d^2+r^2-2drc\right)^{3/2}}dr = \frac{1}{2}\int_0^\infty \frac{2r-2dc+2dc}{\left(d^2+r^2-2drc\right)^{3/2}}dr \tag{19}$$

$$= -\left.\frac{1}{\sqrt{d^2+r^2-2drc}}\right|_0^\infty + dc\int_0^\infty \frac{1}{\left(d^2+r^2-2drc\right)^{3/2}}dr \tag{20}$$

$$= \frac{1}{d} + dc\int_0^\infty \frac{1}{\left(d^2+r^2-2drc\right)^{3/2}}dr \tag{21}$$

We can now complete the square in the remaining integral to get

$$\int_0^\infty \frac{1}{\left(d^2+r^2-2drc\right)^{3/2}}dr = \int_0^\infty \frac{1}{\left((r-dc)^2+(d^2-d^2c^2)\right)^{3/2}}dr \tag{22}$$

$$= \int_0^\infty \frac{1}{\left((r-dc)^2+d^2s^2\right)^{3/2}}dr \tag{23}$$

The last integral is of the form $\int dx/\left(a^2+x^2\right)^{3/2}$ which can be looked up in tables, but just to be complete, let's work that one out too. We can use a trigonometric substitution

$$\begin{aligned} r-dc &= ds\tan u && (24)\\ dr &= ds\sec^2 u\,du && (25)\\ (r-dc)^2+d^2s^2 &= d^2s^2\left(1+\tan^2 u\right) && (26)\\ &= d^2s^2\sec^2 u && (27) \end{aligned}$$

Putting all this into 23 we get (we'll leave the limits off for now):

$$\begin{aligned} \int \frac{1}{\left((r-dc)^2+d^2s^2\right)^{3/2}}dr &= \int \frac{ds\sec^2 u\,du}{d^3s^3\sec^3 u} && (28)\\ &= \frac{1}{d^2s^2}\int \cos u\,du && (29)\\ &= -\frac{1}{d^2s^2}\sin u && (30) \end{aligned}$$

To convert back to r, we use

$$\begin{aligned}
\cos u &= \sqrt{\frac{1}{\sec^2 u}} && (31)\\
&= \sqrt{\frac{1}{1+\tan^2 u}} && (32)\\
&= \sqrt{\frac{1}{1+\frac{(r-dc)^2}{d^2s^2}}} && (33)\\
\sin u &= \pm\sqrt{1-\cos^2 u} && (34)\\
&= \pm\frac{r-dc}{\sqrt{d^2s^2+(r-dc)^2}} && (35)
\end{aligned}$$

so the integral comes out to (restoring the limits):

$$\begin{aligned}
\int_0^\infty \frac{1}{\left((r-dc)^2+d^2s^2\right)^{3/2}}dr &= \pm\frac{1}{d^2s^2}\frac{r-dc}{\sqrt{d^2s^2+(r-dc)^2}}\Bigg|_0^\infty && (36)\\
&= \pm\frac{1}{d^2s^2}(1+c) && (37)
\end{aligned}$$

Since the integrand is positive over its entire range, we must take the + sign in the answer. We can now plug this back into 21 to get

$$\begin{aligned}
\int_0^\infty \frac{r}{(d^2+r^2-2drc)^{3/2}}dr &= \frac{1}{d}+\frac{dc}{d^2s^2}(1+c) && (38)\\
&= \frac{1}{d}\left[1-+\frac{c(1+c)}{1-c^2}\right] && (39)\\
&= \frac{1}{d}\left[1+\frac{c(1+c)}{(1-c)(1+c)}\right] && (40)\\
&= \frac{1}{(1-\cos\theta)d} && (41)
\end{aligned}$$

Plugging this into 16, we now have the remaining integral over θ:

$$
\begin{aligned}
\int_0^\pi \int_0^\infty \frac{r\sin^3\theta}{\left(d^2+r^2-2dr\cos\theta\right)^{3/2}}drd\theta &= \frac{1}{d}\int_0^\pi \frac{\sin^3\theta}{1-\cos\theta}d\theta && (42)\\
&= \frac{1}{d}\int_0^\pi \frac{\sin\theta\left(1-\cos^2\theta\right)}{1-\cos\theta}d\theta && (43)\\
&= \frac{1}{d}\int_0^\pi \frac{\sin\theta\left(1-\cos\theta\right)\left(1+\cos\theta\right)}{1-\cos\theta}d\theta && (44)\\
&= \frac{1}{d}\int_0^\pi \left(\sin\theta+\sin\theta\cos\theta\right)d\theta && (45)\\
&= \frac{2}{d} && (46)
\end{aligned}
$$

Q.E.D.
Now you see why I use Maple to work out integrals!