ANGULAR MOMENTUM CONSERVATION: EXAMPLE WITH A SOLENOID

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.13.

We'll revisit the earlier problem of the two charged cylinders and the solenoid. To reiterate, we have a long solenoid with n turns per unit length carrying current I_0 and a radius R, with its axis along the z axis. The magnetic field inside the solenoid is

$$\mathbf{B}_0 = \mu_0 n I \hat{\mathbf{z}} \tag{1}$$

The field is zero outside the solenoid.

We add two other cylinders (not solenoids), both coaxial with the solenoid. One cylinder has radius $a < R$ (so it lies inside the solenoid) and carries surface charge $+Q$; the other cylinder has radius $b > R$ (outside the solenoid) and carries charge $-Q$. Both cylinders have length ℓ. From Gauss's law, the electric field between these two cylinders is, for $a < r < b$

$$\mathbf{E}_0 = \frac{Q}{2\pi\epsilon_0 \ell}\frac{\hat{\mathbf{r}}}{r} \tag{2}$$

That is, the field points radially outward from the axis. The electric field is zero for $r < a$ and $r > b$. (We're neglecting end effects, so we're assuming that $\ell \gg b > a$.)

In our earlier solution, we worked out the angular momentum contained in the fields and showed that it is equal to the mechanical angular momentum transferred to the two cylinders if the current in the solenoid is slowly reduced. However, there is another effect that we neglected: when the charged cylinders start to rotate, they generate a changing magnetic field inside them which in turn creates a circumferential electric field in the space between the cylinders. When the final rotation speeds of the two cylinders are reached (that is, when the current through the solenoid has been reduced to zero), the cylinders continue rotating, thus generating a static magnetic field that interacts with the electric field to produce an extra amount of angular momentum in the fields. We'll consider this static magnetic field first.

The rotating cylinders are effectively solenoids themselves. The surface charge density on the two cylinders is

$$\sigma_{a,b} = \begin{cases} \frac{Q}{2\pi a\ell} & \text{inner cylinder} \\ -\frac{Q}{2\pi b\ell} & \text{outer cylinder} \end{cases} \tag{3}$$

The surface current densities are

$$K_{a,b} = \begin{cases} \frac{Q(a\omega_a)}{2\pi a\ell} & \text{inner cylinder} \\ -\frac{Q(b\omega_b)}{2\pi b\ell} & \text{outer cylinder} \end{cases} \tag{4}$$

so the magnetic field due to each cylinder is

$$\mathbf{B}_{a,b} = \begin{cases} \frac{\mu_0 Q\omega_a}{2\pi\ell}\hat{\mathbf{z}} & \text{inner cylinder} \\ \frac{\mu_0 Q\omega_b}{2\pi\ell}\hat{\mathbf{z}} & \text{outer cylinder} \end{cases} \tag{5}$$

To get the directions of $\mathbf{B}$ we see from Griffiths's Example 8.4 that the angular momentum of the inner cylinder is in the $+z$ direction and of the outer cylinder is in the $-z$ direction. Since the outer cylinder has *negative* charge, however, its magnetic field points in the *same* direction as that of the inner cylinder. Another way of putting this is to note that the angular velocities are

$$\omega_a = \omega_a\hat{\mathbf{z}} \tag{6}$$
$$\omega_b = -\omega_b\hat{\mathbf{z}} \tag{7}$$

so the minus sign for ω_b cancels the minus sign for the charge on the outer cylinder.

The linear momentum density is non-zero only in the region $a \le r \le b$ and is

$$\mathfrak{p}_{em} = \varepsilon_0 \mathbf{E}\times\mathbf{B} \tag{8}$$
$$= -\frac{\mu_0 Q^2\omega_b}{4\pi^2\ell^2 r}\hat{\phi} \tag{9}$$

The angular momentum density is

$$\mathfrak{L}_{em} = \mathbf{r}\times\mathfrak{p}_{em} \tag{10}$$
$$= -\frac{\mu_0 Q^2\omega_b}{4\pi^2\ell^2}\hat{\mathbf{z}} \tag{11}$$

which is constant, so the total angular momentum is

$$\mathbf{L} = -\frac{\mu_0 Q^2 \omega_b}{4\pi^2 \ell^2}\left[\pi\left(b^2 - a^2\right)\ell\right]\hat{\mathbf{z}} \tag{12}$$

$$= -\frac{\mu_0 Q^2 \omega_b}{4\pi \ell}\left(b^2 - a^2\right)\hat{\mathbf{z}} \tag{13}$$

Now we can look at what's happening as the cylinders are spinning up to their final speeds. As they speed up, the magnetic field due to each cylinder is changing according to

$$\frac{\partial \mathbf{B}_{a,b}}{\partial t} = \begin{cases} \frac{\mu_0 Q \dot{\omega}_a}{2\pi\ell}\hat{\mathbf{z}} & \text{inner cylinder} \\ \frac{\mu_0 Q \dot{\omega}_b}{2\pi\ell}\hat{\mathbf{z}} & \text{outer cylinder} \end{cases} \tag{14}$$

where the dot indicates a time derivative. According to Faraday's law, this changing magnetic field induces a circumferential electric field:

$$\oint \mathbf{E} \cdot d\ell = -\int \frac{\partial \mathbf{B}_{a,b}}{\partial t} \cdot d\mathbf{a} \tag{15}$$

This electric field will exert a torque on each cylinder, whose integral over time will give the angular momentum transferred to the cylinders. First we need to calculate the field at each cylinder. We choose a circular path of integration at the surface of each cylinder. Remember that the field of the inner cylinder is non-zero only for $r < a$, which the the field for the outer cylinder covers the entire region $r < b$.

$$\mathbf{E}_a = -\frac{1}{2\pi a}\pi a^2 \frac{\mu_0 Q\left(\dot{\omega}_a + \dot{\omega}_b\right)}{2\pi\ell}\hat{\phi} \tag{16}$$

$$\mathbf{E}_b = -\frac{1}{2\pi b}\frac{\mu_0 Q\left(\pi a^2 \dot{\omega}_a + \pi b^2 \dot{\omega}_b\right)}{2\pi\ell}\hat{\phi} \tag{17}$$

To confirm the direction of $\mathbf{E}$, recall Lenz's law, which states that the induced field opposes the change that produced it. Since the magnetic field is increasing in the $+z$ direction, the induced electric field must oppose this increase so it must be in the $-\phi$ direction.

The torque on the cylinders is then

$$
\begin{aligned}
\mathbf{N}_a &= \mathbf{r}\times\mathbf{F}_a && (18)\\
&= \mathbf{r}\times Q\mathbf{E}_a && (19)\\
&= a^2\frac{\mu_0 Q^2\left(\dot{\omega}_a+\dot{\omega}_b\right)}{4\pi\ell}\left(-\hat{\mathbf{r}}\times\hat{\phi}\right) && (20)\\
&= -a^2\frac{\mu_0 Q^2\left(\dot{\omega}_a+\dot{\omega}_b\right)}{4\pi\ell}\hat{\mathbf{z}} && (21)\\
\mathbf{N}_b &= -\mathbf{r}\times Q\mathbf{E}_b && (22)\\
&= \frac{\mu_0 Q^2\left(a^2\dot{\omega}_a+b^2\dot{\omega}_b\right)}{4\pi\ell}\hat{\mathbf{z}} && (23)
\end{aligned}
$$

Adding them together we get

$$
\begin{aligned}
\mathbf{N} &= \mathbf{N}_a+\mathbf{N}_b && (24)\\
&= \frac{\mu_0 Q^2\left(b^2-a^2\right)\dot{\omega}_b}{4\pi\ell}\hat{\mathbf{z}} && (25)
\end{aligned}
$$

Integrating over the time it takes to reach the final speed ω_b we get

$$
\mathbf{L}=\frac{\mu_0 Q^2\left(b^2-a^2\right)\omega_b}{4\pi\ell}\hat{\mathbf{z}} \tag{26}
$$

which is equal and opposite to 13. Thus the total angular momentum introduced into the system by magnetic and electric fields induced by the rotating cylinders is zero, showing that angular momentum is conserved.