MOMENTUM OF A POINT CHARGE OUTSIDE A SOLENOID

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 8.14.

Another example of calculating momentum and angular momentum in electromagnetic fields. We have an infinite solenoid along the z axis, of radius R with n turns per unit length and carrying current I. At position $a\hat{\mathbf{x}}$ there is a point charge q. We want to find the momentum and angular momentum of the resulting fields.

The first step in this problem is deciding which coordinate system to use. The solenoid makes us think of cylindrical coordinates, while the point charge suggests spherical. However, these two systems don't mesh well, so we can try the fallback of using rectangular coordinates.

The field of the solenoid is zero outside and uniform inside, where it is

$$\mathbf{B}=\mu_0 nI\hat{\mathbf{z}} \text{ for } x^2+y^2<R^2 \tag{1}$$

If the point charge were located at the origin, its field would be

$$\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{\left(x^2+y^2+z^2\right)^{3/2}} \tag{2}$$

If we shift the charge to $a\hat{\mathbf{x}}$ then we get

$$\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\frac{(x-a)\,\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{\left((x-a)^2+y^2+z^2\right)^{3/2}} \tag{3}$$

The momentum density is

$$\mathfrak{p}_{em}=\varepsilon_0\mathbf{E}\times\mathbf{B} \tag{4}$$

$$=\frac{\mu_0 nIq}{4\pi\left((x-a)^2+y^2+z^2\right)^{3/2}}\left[-(x-a)\,\hat{\mathbf{y}}+y\hat{\mathbf{x}}\right] \tag{5}$$

To get the total momentum we need to integrate this over the interior of the solenoid. It turns out to be easiest to do this by integrating first over z

and then converting to polar coordinates for the remaining two integrations. Integrating over z we get (using Maple or tables)

(6)
$$\frac{\mu_0 nIq}{4\pi}\left[-(x-a)\hat{\mathbf{y}}+y\hat{\mathbf{x}}\right]\int_{-\infty}^{\infty}\frac{dz}{\left((x-a)^2+y^2+z^2\right)^{3/2}} = \frac{\mu_0 nIq}{4\pi}\left[-(x-a)\hat{\mathbf{y}}+y\hat{\mathbf{x}}\right]\frac{2}{(x-a)^2+y^2}$$

$$= \frac{\mu_0 nIq}{2\pi}\frac{(a-x)\hat{\mathbf{y}}+y\hat{\mathbf{x}}}{(x-a)^2+y^2} \tag{7}$$

If we integrate the $\hat{\mathbf{x}}$ component over y we have

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\frac{y\hat{\mathbf{x}}}{(x-a)^2+y^2}=0 \tag{8}$$

because the integrand is an odd function of y and the integral is over a symmetric interval. This leaves us with the $\hat{\mathbf{y}}$ component, and it is here that we turn to polar coordinates. Using $x=r\cos\theta$ and $y=r\sin\theta$ we have

$$\int_0^R\int_0^{2\pi}\frac{(a-x)\hat{\mathbf{y}}}{(x-a)^2+y^2}r\,d\theta\,dr = \int_0^R\int_0^{2\pi}\frac{(a-r\cos\theta)r\hat{\mathbf{y}}}{r^2-2ar\cos\theta+a^2}d\theta\,dr \tag{9}$$

$$= \hat{\mathbf{y}}\int_0^R\frac{2\pi r}{a}dr \tag{10}$$

$$= \frac{\pi R^2}{a}\hat{\mathbf{y}} \tag{11}$$

where we used Maple to do the θ integral. The total momentum is thus

$$\mathbf{P}_{em} = \frac{\mu_0 nIq}{2\pi}\frac{\pi R^2}{a}\hat{\mathbf{y}} = \frac{\mu_0 nIqR^2}{2a}\hat{\mathbf{y}} \tag{12}$$

The angular momentum density is

(13)
$$\mathfrak{L}_{em} = \mathbf{r} \times \mathfrak{p}_{em}$$
(14)
$$= \frac{\mu_0 n I q}{4\pi\left((x-a)^2 + y^2 + z^2\right)^{3/2}}\left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right] \times \left[-(x-a)\hat{\mathbf{y}} + y\hat{\mathbf{x}}\right]$$
(15)
$$= \frac{\mu_0 n I q}{4\pi\left((x-a)^2 + y^2 + z^2\right)^{3/2}}\left[z(x-a)\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + \left(x(a-x) - y^2\right)\hat{\mathbf{z}}\right]$$

Integrating the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components over z gives zero because the integrand is an odd function of z integrated over a symmetric interval. Thus we are left with the $\hat{\mathbf{z}}$ component which we can again integrate first over z and then over the other two coordinates using polar coordinates. We have

(16)
$$\mathbf{L}_{em} = \frac{\mu_0 n I q}{4\pi}\hat{\mathbf{z}}\int_0^R\int_0^{2\pi}\int_{-\infty}^{\infty} dz \frac{\left(x(a-x) - y^2\right) r\, d\theta\, dr}{\left((x-a)^2 + y^2 + z^2\right)^{3/2}}$$
(17)
$$= \frac{\mu_0 n I q}{4\pi}\hat{\mathbf{z}}\int_0^R\int_0^{2\pi}\frac{2\left(x(a-x) - y^2\right)}{(x-a)^2 + y^2} r\, d\theta\, dr$$
(18)
$$= \frac{\mu_0 n I q}{2\pi}\hat{\mathbf{z}}\int_0^R\int_0^{2\pi}\frac{\left(r\cos\theta\,(a - r\cos\theta) - r^2\sin^2\theta\right) r}{r^2 - 2ar\cos\theta + a^2} d\theta\, dr$$
(19)
$$= \frac{\mu_0 n I q}{2\pi}\hat{\mathbf{z}}\int_0^R\int_0^{2\pi}\frac{\left(ar\cos\theta - r^2\right) r}{r^2 - 2ar\cos\theta + a^2} d\theta\, dr$$
(20)
$$= \frac{\mu_0 n I q}{2\pi}\hat{\mathbf{z}}\int_0^R\left[\frac{2\pi r^2}{a^2 - r^2} - \frac{2\pi r^2}{a^2 - r^2}\right] r\, dr$$
(21)
$$= 0$$

where again we used Maple to do the integral.