WAVE EQUATION: DERIVATION AND EXAMPLES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.1.

As a prelude to the study of electromagnetic waves, we'll have a look at a derivation of the wave equation.

Waves can appear in any form of matter, as well as in electromagnetic fields, so we'll look at the easiest case for a derivation. Suppose we have a string under a tension T. Initially, we'll stretch out the string along the z axis so that it's perfectly straight. Now suppose we pull the string slightly to one side (in the x direction, say) so that the string now follows a slightly curved path. Suppose that at position z the tangent to the string makes an angle θ with the z axis and at a slightly different position $z+dz$ the angle is $\theta+d\theta$. The tension can be resolved into an x component (perpendicular to the original orientation of the string) and a z component (parallel to the original orientation). We get, at position z:

$$T_x(z) = T\sin\theta \tag{0.1}$$
$$T_z(z) = T\cos\theta \tag{0.2}$$

and at position $z+dz$:

$$T_x(z+dz) = T\sin(\theta+d\theta) \tag{0.3}$$
$$T_z(z+dz) = T\cos(\theta+d\theta) \tag{0.4}$$

For small angles, the sine and tangent are equal to first order:

$$\sin\theta \approx \theta \approx \tan\theta \tag{0.5}$$

and the tangent is the slope of tangent to the string. Since the string was pulled back in the x direction, x is the displacement, so

$$\tan\theta = \frac{\partial x}{\partial z} \tag{0.6}$$

and the difference in the x component of the tension between the two points on the string is

$$T_x(z+dz)-T_x(z) \approx T\tan(\theta+d\theta)-T\tan\theta \tag{0.7}$$

$$= T\left(\left.\frac{\partial x}{\partial z}\right|_{z+dz}-\left.\frac{\partial x}{\partial z}\right|_{z}\right) \tag{0.8}$$

The second derivative is defined as

$$\frac{\partial^2 x}{\partial z^2}=\lim_{dz\to 0}\frac{1}{dz}\left(\left.\frac{\partial x}{\partial z}\right|_{z+dz}-\left.\frac{\partial x}{\partial z}\right|_{z}\right) \tag{0.9}$$

so the difference in T_x is, in the limit $dz\to 0$

$$dT_x=\frac{\partial^2 x}{\partial z^2}T\,dz \tag{0.10}$$

However, from Newton's law $F=ma$, the net force on the string segment is also equal to the mass of that string segment times its acceleration. If the mass per unit length is μ, then

$$dT_x=\mu\,dz\frac{\partial^2 x}{\partial t^2} \tag{0.11}$$

Equating these last two formulas gives us the wave equation

$$\frac{\partial^2 x}{\partial z^2}=\frac{\mu}{T}\frac{\partial^2 x}{\partial t^2} \tag{0.12}$$

The coefficient μ/T has the units of mass divided by force, which is velocity squared, so we can define a velocity

$$v\equiv\sqrt{\frac{T}{\mu}} \tag{0.13}$$

and write the wave equation as

$$\boxed{\frac{\partial^2 x}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 x}{\partial t^2}} \tag{0.14}$$

Although we haven't shown it here, it turns out that v is the speed of propagation of the wave.

It turns out that any function of form

$$x = f(z \pm vt) \tag{0.15}$$

is a solution of the wave equation as can readily be seen by direct differentiation.

Example 1. With

$$f = Ae^{-b(z-vt)^2} \tag{0.16}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = 2bAe^{-b(z-vt)^2}\left(2bv^2t^2 - 4bzvt + 2bz^2 - 1\right) \tag{0.17}$$

$$\frac{\partial^2 x}{\partial t^2} = 2bAv^2e^{-b(z-vt)^2}\left(2bv^2t^2 - 4bzvt + 2bz^2 - 1\right) = v^2\frac{\partial^2 x}{\partial z^2} \tag{0.18}$$

so 0.14 is satisfied.

Example 2. With

$$f = A\sin(b(z-vt)) \tag{0.19}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = -Ab^2\sin(b(z-vt)) \tag{0.20}$$

$$\frac{\partial^2 x}{\partial t^2} = -Ab^2v^2\sin(b(z-vt)) = v^2\frac{\partial^2 x}{\partial z^2} \tag{0.21}$$

Example 3. With

$$f = \frac{A}{b(z-vt)^2+1} \tag{0.22}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = \frac{8Ab^2(z-vt)^2}{\left(b(z-vt)^2+1\right)^3} - \frac{2Ab}{\left(b(z-vt)^2+1\right)^2} \tag{0.23}$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{8Ab^2v^2(z-vt)^2}{\left(b(z-vt)^2+1\right)^3} - \frac{2Abv^2}{\left(b(z-vt)^2+1\right)^2} = v^2\frac{\partial^2 x}{\partial z^2} \tag{0.24}$$

Functions that don't have the form 0.15 don't satisfy the wave equation.

Example 4. With

$$f = Ae^{-b\left(bz^2+vt\right)} \tag{0.25}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = 2Ab^2e^{-b\left(bz^2+vt\right)}\left(2b^2z^2-1\right) \tag{0.26}$$

$$\frac{\partial^2 x}{\partial t^2} = Ab^2v^2e^{-b\left(bz^2+vt\right)} \neq v^2\frac{\partial^2 x}{\partial z^2} \tag{0.27}$$

Example 5. With

$$f = A\sin(bz)\cos(bvt)^3 \tag{0.28}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = -Ab^2\sin(bz)\cos(bvt)^3 \tag{0.29}$$

$$\frac{\partial^2 x}{\partial t^2} = -3A\sin(bz)b^3v^3t\left(3(bvt)^3\cos(bvt)^3+2\sin(bvt)^3\right) \neq v^2\frac{\partial^2 x}{\partial z^2} \tag{0.30}$$

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