

WAVE EQUATION: DERIVATION AND EXAMPLES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.1.

As a prelude to the study of electromagnetic waves, we'll have a look at a derivation of the wave equation.

Waves can appear in any form of matter, as well as in electromagnetic fields, so we'll look at the easiest case for a derivation. Suppose we have a string under a tension T . Initially, we'll stretch out the string along the z axis so that it's perfectly straight. Now suppose we pull the string slightly to one side (in the x direction, say) so that the string now follows a slightly curved path. Suppose that at position z the tangent to the string makes an angle θ with the z axis and at a slightly different position $z + dz$ the angle is $\theta + d\theta$. The tension can be resolved into an x component (perpendicular to the original orientation of the string) and a z component (parallel to the original orientation). We get, at position z :

$$(1) \quad T_x(z) = T \sin \theta$$

$$(2) \quad T_z(z) = T \cos \theta$$

and at position $z + dz$:

$$(3) \quad T_x(z + dz) = T \sin(\theta + d\theta)$$

$$(4) \quad T_z(z + dz) = T \cos(\theta + d\theta)$$

For small angles, the sine and tangent are equal to first order:

$$(5) \quad \sin \theta \approx \theta \approx \tan \theta$$

and the tangent is the slope of tangent to the string. Since the string was pulled back in the x direction, x is the displacement, so

$$(6) \quad \tan \theta = \frac{\partial x}{\partial z}$$

and the difference in the x component of the tension between the two points on the string is

$$(7) \quad T_x(z+dz) - T_x(z) \approx T \tan(\theta + d\theta) - T \tan \theta$$

$$(8) \quad = T \left(\left. \frac{\partial x}{\partial z} \right|_{z+dz} - \left. \frac{\partial x}{\partial z} \right|_z \right)$$

The second derivative is defined as

$$(9) \quad \frac{\partial^2 x}{\partial z^2} = \lim_{dz \rightarrow 0} \frac{1}{dz} \left(\left. \frac{\partial x}{\partial z} \right|_{z+dz} - \left. \frac{\partial x}{\partial z} \right|_z \right)$$

so the difference in T_x is, in the limit $dz \rightarrow 0$

$$(10) \quad dT_x = \frac{\partial^2 x}{\partial z^2} T dz$$

However, from Newton's law $F = ma$, the net force on the string segment is also equal to the mass of that string segment times its acceleration. If the mass per unit length is μ , then

$$(11) \quad dT_x = \mu dz \frac{\partial^2 x}{\partial t^2}$$

Equating these last two formulas gives us the wave equation

$$(12) \quad \frac{\partial^2 x}{\partial z^2} = \frac{\mu}{T} \frac{\partial^2 x}{\partial t^2}$$

The coefficient μ/T has the units of mass divided by force, which is velocity squared, so we can define a velocity

$$(13) \quad v \equiv \sqrt{\frac{T}{\mu}}$$

and write the wave equation as

$$(14) \quad \boxed{\frac{\partial^2 x}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 x}{\partial t^2}}$$

Although we haven't shown it here, it turns out that v is the speed of propagation of the wave.

It turns out that any function of form

$$(15) \quad x = f(z \pm vt)$$

is a solution of the wave equation as can readily be seen by direct differentiation.

Example 1. With

$$(16) \quad f = Ae^{-b(z-vt)^2}$$

we have

$$(17) \quad \frac{\partial^2 x}{\partial z^2} = 2bAe^{-b(z-vt)^2} (2bv^2t^2 - 4bzvt + 2bz^2 - 1)$$

$$(18) \quad \frac{\partial^2 x}{\partial t^2} = 2bAv^2e^{-b(z-vt)^2} (2bv^2t^2 - 4bzvt + 2bz^2 - 1) = v^2 \frac{\partial^2 x}{\partial z^2}$$

so 14 is satisfied.

Example 2. With

$$(19) \quad f = A \sin(b(z - vt))$$

we have

$$(20) \quad \frac{\partial^2 x}{\partial z^2} = -Ab^2 \sin(b(z - vt))$$

$$(21) \quad \frac{\partial^2 x}{\partial t^2} = -Ab^2v^2 \sin(b(z - vt)) = v^2 \frac{\partial^2 x}{\partial z^2}$$

Example 3. With

$$(22) \quad f = \frac{A}{b(z - vt)^2 + 1}$$

we have

$$(23) \quad \frac{\partial^2 x}{\partial z^2} = \frac{8Ab^2(z-vt)^2}{(b(z-vt)^2+1)^3} - \frac{2Ab}{(b(z-vt)^2+1)^2}$$

$$(24) \quad \frac{\partial^2 x}{\partial t^2} = \frac{8Ab^2v^2(z-vt)^2}{(b(z-vt)^2+1)^3} - \frac{2Abv^2}{(b(z-vt)^2+1)^2} = v^2 \frac{\partial^2 x}{\partial z^2}$$

Functions that don't have the form 15 don't satisfy the wave equation.

Example 4. With

$$(25) \quad f = Ae^{-b(bz^2+vt)}$$

we have

$$(26) \quad \frac{\partial^2 x}{\partial z^2} = 2Ab^2e^{-b(bz^2+vt)}(2b^2z^2-1)$$

$$(27) \quad \frac{\partial^2 x}{\partial t^2} = Ab^2v^2e^{-b(bz^2+vt)} \neq v^2 \frac{\partial^2 x}{\partial z^2}$$

Example 5. With

$$(28) \quad f = A \sin(bz) \cos(bvt)^3$$

we have

$$(29) \quad \frac{\partial^2 x}{\partial z^2} = -Ab^2 \sin(bz) \cos(bvt)^3$$

$$(30) \quad \frac{\partial^2 x}{\partial t^2} = -3A \sin(bz) b^3 v^3 t \left(3(bvt)^3 \cos(bvt)^3 + 2 \sin(bvt)^3 \right) \neq v^2 \frac{\partial^2 x}{\partial z^2}$$

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