WAVE EQUATION: WAVE SPEED AND STANDING WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.2.

Any function of form $f = f(z \pm vt)$ (for a constant v) is a solution of the wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{1}$$

Once we specify f, we can plot it as a function of z for a fixed value of t, say t_1. This shows a snapshot of the wave at a particular time. If we then plot f as a function of z at some later time t_2, the wave has the same shape but is shifted along the z axis. A point at which f has the value $f_1 \equiv f(z_1 - vt_1)$ will have moved to a position z_2 where $z_2 - vt_2 = z_1 - vt_1$. In other words, that particular point on the wave moves from z_1 to z_2 in a time $t_2 - t_1$, and must be moving with speed

$$\frac{z_2 - z_1}{t_2 - t_1} = \frac{v(t_2 - t_1)}{t_2 - t_1} = v \tag{2}$$

Thus v is the speed of the wave. If $z_2 > z_1$, then $v > 0$ and the wave moves towards increasing z. If $z_2 < z_1$, then $v < 0$ and the wave moves towards decreasing z.

Since the wave equation is linear, any linear combination of solutions is also a solution, so we can add together waves with equal and opposite speeds. For example, if we have two sine waves with equal and opposite speeds, we can add them together to get

$$\begin{aligned} f(z,t) = \sin(k(z - vt)) + \sin(k(z + vt)) &= \sin kz \cos kvt - \cos kz \sin kvt + \qquad (3)\\ & \quad \sin kz \cos kvt + \cos kz \sin kvt \\ &= 2 \sin kz \cos kvt \qquad (4) \end{aligned}$$

where k is a constant.

The latter form is not a function of form $f(z \pm vt)$ but as it's a sum of two functions of this type, it is still a solution of the wave equation. In fact, it represents a standing wave, that is, a wave which doesn't travel along the z axis, but rather oscillates in place. We can see this since for all points

$z_n = n\pi/k$ where n is an integer, $f(z_n,t) = 0$ for all times t. For all times $t_n = (2n+1)\pi/2kv$, $f(z,t_n) = 0$ for all positions, so the wave becomes a flat line. In between these times, the wave oscillates between its maximum and minimum heights of $\pm 2\sin kz$.

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