WAVE EQUATION: SINUSOIDAL WAVES AND COMPLEX NOTATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.3.

Any function of form $f = f(z \pm vt)$ (for a constant v) is a solution of the wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{1}$$

Probably the most commonly used solution is the sine wave, which can be written as either a sine or a cosine, either of which gives the most general sinusoidal solution. For example, we can write

$$f(z,t) = A\cos(k(z - vt) + \delta) \tag{2}$$

where k, v and δ are constants. The wavelength λ is the distance (at constant time) between two successive peaks, which occurs when z advances far enough to increase the argument of the cosine by 2π. Therefore

$$k\lambda = 2\pi \tag{3}$$

$$\lambda = \frac{2\pi}{k} \tag{4}$$

The parameter k is called the *wave number*.

As we saw earlier, v is the speed of the wave. The parameter δ is the phase of the wave, and allows for waves that don't have their maximum at $z - vt = 0$. Suppose that at $t = 0$ the maximum of the wave that is closest to $z = 0$ is at $z = z_m$. Then the argument of the cosine must be zero at that point, so

$$kz_m = -\delta \tag{5}$$

$$z_m = -\frac{\delta}{k} \tag{6}$$

Thus a positive phase ($\delta > 0$) means that the peak of the wave lags behind $z = 0$, while a negative phase ($\delta < 0$) means the peak leads $z = 0$.

It's more common to write the wave in terms of k and the angular frequency, defined as

$$\omega \equiv kv = \frac{2\pi v}{\lambda} \tag{7}$$

so we get

$$f(z,t) = A\cos(kz - \omega t + \delta) \tag{8}$$

It turns out that it's often more convenient to represent a sinusoidal wave as the real part of a complex exponential. That is

$$f(z,t) = \Re\left(Ae^{i(kz-\omega t+\delta)}\right) \tag{9}$$

We can then define a complex wave function

$$\begin{aligned} \tilde{f}(z,t) &\equiv \tilde{A}e^{i(kz-\omega t)} && (10)\\ \tilde{A} &\equiv Ae^{i\delta} && (11) \end{aligned}$$

Physical waves are, of course, always real functions (well, except in quantum mechanics, where complex wave functions are the norm, but even there, any physical interpretation of the quantum wave function requires extracting a real value from the complex function).

Working with complex exponentials is, in most cases, easier than working with sines and cosines, although the formulas aren't quite trivial. For example, if we want to add two waves f_1 and f_2 then we get

$$\begin{aligned} f_3 &= f_1 + f_2 && (12)\\ &= \Re\left(\tilde{f}_1 + \tilde{f}_2\right) && (13) \end{aligned}$$

If the two waves to be added have the same k and ω, then we need to find the new amplitude A_3 and phase δ_3 in terms of the amplitudes and phases of the constituent waves. That is

$$\begin{aligned} \tilde{f}_3 &= \tilde{A}_3 e^{i(kz-\omega t)} && (14)\\ A_3 e^{i\delta_3} e^{i(kz-\omega t)} &= \left(\tilde{A}_1 + \tilde{A}_2\right) e^{i(kz-\omega t)} && (15)\\ A_3 e^{i\delta_3} &= A_1 e^{i\delta_1} + A_2 e^{i\delta_2} && (16) \end{aligned}$$

Taking the square modulus of both sides, we get

$$A_3^2 = A_1^2 + A_2^2 + A_1 A_2 \left(e^{i(\delta_1 - \delta_2)} + e^{-i(\delta_1 - \delta_2)} \right) \tag{17}$$

$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos(\delta_1 - \delta_2) \tag{18}$$

$$A_3 = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\delta_1 - \delta_2)} \tag{19}$$

To get the phase, we can use

$$\tilde{A}_3 = A_3 (\cos\delta_3 + i \sin\delta_3) \tag{20}$$

$$\tan\delta_3 = \frac{\Im \tilde{A}_3}{\Re \tilde{A}_3} \tag{21}$$

$$= \frac{\Im \left(\tilde{A}_1 + \tilde{A}_2\right)}{\Re \left(\tilde{A}_1 + \tilde{A}_2\right)} \tag{22}$$

$$= \frac{A_1 \sin\delta_1 + A_2 \sin\delta_2}{A_1 \cos\delta_1 + A_2 \cos\delta_2} \tag{23}$$

$$\delta_3 = \arctan\left[\frac{A_1 \sin\delta_1 + A_2 \sin\delta_2}{A_1 \cos\delta_1 + A_2 \cos\delta_2}\right] \tag{24}$$

Keep in mind that these formulas work only for adding two waves with the *same* wave number k and frequency ω.

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