WAVES: BOUNDARY CONDITIONS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.5.

For a one dimensional wave (on a string, say) suppose we now place a boundary at the point $z=0$. For a string, this could be a point at which one string is joined with another string of a different mass per unit length. If we send a wave down the string from large negative z, when this wave reaches $z=0$, there will be a reflected wave that returns towards negative z and a transmitted wave that proceeds beyond $z=0$ towards positive z. We can get some idea of the nature of these reflected and transmitted waves if we impose some boundary conditions at the point $z=0$.

First, we require the wave function f to be continuous, for the simple reason that there is no break in the string at $z=0$. The second boundary condition requires that the derivative $\partial f/\partial z$ is also continuous. The reason for this is a bit more subtle. Assuming there is no point mass (such as a knot) at the joining position, if the tangent to the string at that point were not continuous, then the second derivative would be infinite, meaning that there would be an infinite force at that point.

To see how the incident, reflected and transmitted waves are related, suppose we have an incident wave $I(z-v_1t)$, a reflected wave $R(z+v_1t)$ and a transmitted wave $T(x-v_2t)$, where I and T are moving to the right, with I defined for $z<0$ and T for $z>0$, and R moving to the left for $z<0$.

The continuity of the wave function at $z=0$ gives us

$$I(-v_1t)+R(v_1t)=T(-v_2t) \tag{1}$$

The continuity of the derivative, if applied directly, just gives the same equation with each function replaced by its derivative, so doesn't help much:

$$\left.\frac{\partial I}{\partial z}\right|_{z=0^-}+\left.\frac{\partial R}{\partial z}\right|_{z=0^-}=\left.\frac{\partial T}{\partial z}\right|_{z=0^+} \tag{2}$$

However, if we consider the original definition of a derivative as a limit, we can make some progress. Consider first the derivative of the incident wave just below $z=0$, at time $t=0$:

$$\left.\frac{\partial I}{\partial z}\right|_{z=0^-}=\lim_{\Delta z\to 0}\frac{I(0)-I(-\Delta z)}{\Delta z} \tag{3}$$

The wave amplitude at the point $(z,t) = (-\Delta z, 0)$ will be at $z = 0$ after it travels the distance Δz, which takes a time $t = \Delta z / v_1$.

By a similar argument, the derivative of the reflected wave is

$$\left.\frac{\partial R}{\partial z}\right|_{z=0^-} = \lim_{\Delta z \to 0} \frac{R(0) - R(-\Delta z)}{\Delta z} \tag{4}$$

This time, the wave amplitude at $(-\Delta z, 0)$ was at $z = 0$ at time $t = -\Delta z / v_1$ since this wave is travelling to the left. Finally, for the transmitted wave

$$\left.\frac{\partial T}{\partial z}\right|_{z=0^+} = \lim_{\Delta z \to 0} \frac{T(\Delta z) - T(0)}{\Delta z} \tag{5}$$

since this wave is defined for $z > 0$. The wave amplitude at $(\Delta z, 0)$ was at $z = 0$ at time $t = -\Delta z / v_2$ since the transmitted wave is travelling to the right with speed v_2.

We can now use the continuity condition 1 to eliminate either R or T from the limits. Start by eliminating R by evaluating everything at time $t = -\Delta z / v_1$:

$$\lim_{\Delta z \to 0} \frac{R(0) - R(-\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[T(0) - I(0) - \left(T\left(\frac{v_2}{v_1}\Delta z\right) - I(\Delta z)\right)\right] \tag{6}$$

Now we can insert this into the continuity equation for derivatives 2, and we'll leave off the limit and $1/\Delta z$ to simplify the notation:

$$I(0) - I(-\Delta z) + T(0) - I(0) - \left(T\left(\frac{v_2}{v_1}\Delta z\right) - I(\Delta z)\right) = T(\Delta z) - T(0) \tag{7}$$

$$(I(0) - I(-\Delta z)) + (I(\Delta z) - I(0)) = \left(T\left(\frac{v_2}{v_1}\Delta z\right) - T(0)\right) + (T(\Delta z) - T(0) \tag{8}$$

Restoring the limit and $1/\Delta z$ we get

$$2\frac{\partial I}{\partial z} = \left(\frac{v_2}{v_1} + 1\right)\frac{\partial T}{\partial z} \tag{9}$$

This condition is strictly true only at $z = 0$, but it must be true for all times. We can convert the derivatives into time derivatives by noting that since $I = I(z - v_1 t)$ we have

$$\frac{\partial I}{\partial z}=-\frac{1}{v_1}\frac{\partial I}{\partial t} \tag{10}$$

Similarly for R and T:

$$\begin{aligned}\frac{\partial T}{\partial z} &= -\frac{1}{v_2}\frac{\partial T}{\partial t} &(11)\\ \frac{\partial R}{\partial z} &= \frac{1}{v_1}\frac{\partial R}{\partial t} &(12)\end{aligned}$$

Returning to 9 we get

$$\begin{aligned}-\frac{2}{v_1}\frac{\partial I}{\partial t} &= -\frac{1}{v_2}\left(\frac{v_2}{v_1}+1\right)\frac{\partial T}{\partial t} &(13)\\ \frac{\partial T}{\partial t} &= \frac{2v_2}{v_1+v_2}\frac{\partial I}{\partial t} &(14)\end{aligned}$$

At $z=0$, we can integrate with respect to time to get

$$T(-v_2 t)=\frac{2v_2}{v_1+v_2}I(-v_1 t)+K_T \tag{15}$$

where K_T is a constant of integration. Although I, T and R are functions of both z and t, they are all actually functions of only one variable, since z and t must always occur in the combination $z\pm v_{1,2}t$. Thus what 15 is saying is that, if we have the incident wave in the form $I(u)$, then the transmitted wave has the form

$$T\left(\frac{v_2}{v_1}u\right)=\frac{2v_2}{v_1+v_2}I(u)+K_T \tag{16}$$

or, conversely

$$T(u)=\frac{2v_2}{v_1+v_2}I\left(\frac{v_2}{v_1}u\right)+K_T \tag{17}$$

For $T(u)$, $u=z-v_2 t$ for $z\geq 0$ (and all times t), while for $I\left(\frac{v_2}{v_1}u\right)$, $u=z-v_1 t$ for $z\leq 0$. If we pick a particular numerical value for u, say 42, then we can write

$$\begin{aligned}T(z_T-v_2 t_T) &= \frac{2v_2}{v_1+v_2}I\left(\frac{v_2}{v_1}(z_I-v_1 t_I)\right)+K_T &(18)\\ T(42) &= \frac{2v_2}{v_1+v_2}I\left(42\frac{v_2}{v_1}\right)+K_T &(19)\end{aligned}$$

and this equation is valid for *all* values of z and t such that

$$z_T - v_2 t_T = 42 \ (z \geq 0) \tag{20}$$

$$z_I - v_1 t_I = 42 \ (z \leq 0) \tag{21}$$

That is, the values of z_T and z_I need not be equal, and neither must $t_T = t_I$. All that matters is that $z_T - v_2 t_T = z_I - v_1 t_I$.

For the reflected wave, we eliminate T using 1 at time $t = -\Delta z/v_2$:

$$\lim_{\Delta z\to 0}\frac{T(\Delta z)-T(0)}{\Delta z} = \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left[I\left(\frac{v_1}{v_2}\Delta z\right)+R\left(-\frac{v_1}{v_2}\Delta z\right)-I(0)-R(0)\right] \tag{22}$$

Substitute into 2, again without the limit and $1/\Delta z$ to simplify the notation:

$$I(0)-I(-\Delta z)+R(0)-R(-\Delta z) = I\left(\frac{v_1}{v_2}\Delta z\right)+R\left(-\frac{v_1}{v_2}\Delta z\right)-I(0)-R(0) \tag{23}$$

$$(I(0)-I(-\Delta z))-\left(I\left(\frac{v_1}{v_2}\Delta z\right)-I(0)\right) = -\left(R(0)-R\left(-\frac{v_1}{v_2}\Delta z\right)\right)-(R(0)-R(-\Delta z)) \tag{24}$$

Restoring the limit and $1/\Delta z$ we get

$$\left(1-\frac{v_1}{v_2}\right)\frac{\partial I}{\partial z} = -\left(1+\frac{v_1}{v_2}\right)\frac{\partial R}{\partial z} \tag{25}$$

$$-\frac{1}{v_1}\left(1-\frac{v_1}{v_2}\right)\frac{\partial I}{\partial t} = -\frac{1}{v_1}\left(1+\frac{v_1}{v_2}\right)\frac{\partial R}{\partial t} \tag{26}$$

$$R(+v_1 t) = \frac{v_2-v_1}{v_1+v_2}I(-v_1 t) \tag{27}$$

$$R(u) = \frac{v_2-v_1}{v_1+v_2}I(-u) \tag{28}$$

Again, for R, $u = z + v_1 t$ with $z \leq 0$ and for I, $u = z - v_1 t$ with $z \leq 0$. These results apply to any wave shape, not just to sinusoidal waves.

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