WAVES: BOUNDARY CONDITION WITH A MASSIVE KNOT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.6.

In our earlier analysis of the boundary conditions for a one dimensional wave, we looked at the case of one string of mass per unit length μ_1 joined at $z=0$ to another string with mass density μ_2, and we assumed that the knot joining the two strings was massless. This led to the boundary conditions being that both the wave function and its first derivative are continuous at $z=0$.

Suppose we now take the knot to have a non-zero mass m. In that case, the derivative no longer needs to be continuous at the knot. We can analyze the situation the same way as we derived the original wave equation. Since the tensions on either side of the knot are equal in magnitude but can now point in different directions, we have the net transverse force on the knot

$$F_t = T\left(\left.\frac{\partial f}{\partial z}\right|_{0^+} - \left.\frac{\partial f}{\partial z}\right|_{0^-}\right) \tag{1}$$

where T is the tension in the strings.

This force is equal to the knot's mass times its transverse acceleration, so

$$T\left(\left.\frac{\partial f}{\partial z}\right|_{0^+} - \left.\frac{\partial f}{\partial z}\right|_{0^-}\right) = m\left.\frac{\partial^2 f}{\partial t^2}\right|_0 \tag{2}$$

Now suppose we have an incident sinusoidal wave

$$I(z,t) = A_I e^{i(k_1 z - \omega t)} \tag{3}$$

using complex notation, so A_I is a complex amplitude with real magnitude a_I and phase δ_1:

$$A_I = a_I e^{i\delta_1} \tag{4}$$

(I'll drop the tildes that Griffiths uses in his book, since we'll be dealing with complex notation pretty well exclusively until we extract the physical meaning at the end.)

When this wave hits the knot, part of it is reflected and part is transmitted, so the overall wave function is

$$f(z,t)=\begin{cases}A_Ie^{i(k_1z-\omega t)}+A_Re^{i(-k_1z-\omega t)} & z<0\\ A_Te^{i(k_2z-\omega t)} & z>0\end{cases} \tag{5}$$

The frequency ω is the same on both sides of $z=0$, but the wave numbers k_1 and k_2 (and thus the wave speeds $v_i=\omega/k_i$) are different. The continuity condition gives us

$$A_I+A_R=A_T \tag{6}$$

and the derivative condition 2 gives us

$$ik_2A_T-ik_1(A_I-A_R)=-\frac{m\omega^2}{T}A_T \tag{7}$$

These two equations can be solved to give A_R and A_T in terms of A_I.

Example. Suppose the string for $z<0$ has a non-zero mass μ_1 so the wave's speed v_1 and wave number k_1 are both finite and non zero. However, the string for $z>0$ is taken to be very light (effectively massless). In this case, the speed is (as we saw in the derivation of the wave equation)

$$v_2=\sqrt{\frac{T}{\mu_2}}=\infty \tag{8}$$

so

$$k_2=\frac{\omega}{v_2}=0 \tag{9}$$

Then from 7 and 6 we have

$$\frac{Tk_1}{m\omega^2}i(A_I-A_R)=A_T=A_I+A_R \tag{10}$$

With the definition

$$\alpha\equiv\frac{Tk_1}{m\omega^2} \tag{11}$$

we can solve this to get

$$A_R = A_I\frac{\alpha i-1}{1+\alpha i} \tag{12}$$

$$A_T = A_I\frac{2\alpha i}{1+\alpha i} \tag{13}$$

Taking magnitudes we get the real amplitudes of the reflected and transmitted waves:

$$a_R = a_I \tag{14}$$

$$a_T = \frac{2\alpha}{\sqrt{1+\alpha^2}}a_I \tag{15}$$

To get the phases, we can find the ratio of the imaginary and real parts in 12 and 13 to get the tangents of the phases. This is a bit messy so it's easiest to use Maple to do the algebra. We get

$$\tan\delta_R = \frac{2\alpha\cos\delta_1 + \left(\alpha^2-1\right)\sin\delta_1}{(\alpha^2-1)\cos\delta_1 - 2\alpha\sin\delta_1} \tag{16}$$

$$= \frac{2\alpha + \left(\alpha^2-1\right)\tan\delta_1}{(\alpha^2-1) - 2\alpha\tan\delta_1} \tag{17}$$

$$= \frac{\frac{2\alpha}{\alpha^2-1} + \tan\delta_1}{1 - \frac{2\alpha}{\alpha^2-1}\tan\delta_1} \tag{18}$$

We can simplify this equation by using the formula for the tangent of the sum of two angles:

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \tag{19}$$

We get

$$\tan\delta_R = \tan\left(\arctan\frac{2\alpha}{\alpha^2-1} + \delta_1\right) \tag{20}$$

$$\delta_R = \arctan\frac{2\alpha}{\alpha^2-1} + \delta_1 \tag{21}$$

For the transmitted wave, we have

$$\tan\delta_T = \frac{\cos\delta_1 + \alpha\sin\delta_1}{-\sin\delta_1 + \alpha\cos\delta_1} \tag{22}$$

$$= \frac{\frac{1}{\alpha} + \tan\delta_1}{1 - \frac{1}{\alpha}\tan\delta_1} \tag{23}$$

$$= \tan\left(\arctan\frac{1}{\alpha} + \delta_1\right) \tag{24}$$

$$\delta_T = \arctan\frac{1}{\alpha} + \delta_1 \tag{25}$$

It might seem odd that the reflected wave has the same amplitude as the incident wave, yet the transmitted wave still has a non-zero amplitude. Doesn't this violate conservation of energy, since the kinetic energy of the string due to the reflected wave is equal to that of the incident wave, so how can there be any energy left over for a transmitted wave? The answer is that since the transmitted wave is in a massless string, it carries no energy, so energy can still be conserved.