WAVES IN A VISCOUS FLUID

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.7.

Another example of a transverse wave is that of a string embedded in a viscous fluid which adds a drag force on the string. The drag force is proportional to the string's transverse speed, so we need to insert a term

$$\Delta F_{drag} = -\gamma \frac{\partial f}{\partial t} \Delta z \tag{1}$$

where γ is a constant determined by the viscosity of the fluid, into our derivation of the original wave equation. This gives us a modified wave equation:

$$T \frac{\partial^2 f}{\partial z^2} - \gamma \frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial t^2} \tag{2}$$

where T is the tension in the string and μ is the mass per unit length of the string.

If we assume that string's frequency ω is constant, then we can write

$$f(z,t) = e^{-i\omega t} F(z) \tag{3}$$

and the wave equation reduces to an ODE:

$$\begin{aligned} TF'' + i\omega\gamma F &= -\mu\omega^2 F && (4) \\ F'' &= -\frac{i\omega\gamma + \mu\omega^2}{T} F \equiv \alpha^2 F && (5) \end{aligned}$$

The general solution is

$$F(z) = Ae^{\alpha z} + Be^{-\alpha z} \tag{6}$$

However, α is a complex number, so F contains both oscillatory and exponential parts.

Using Maple, we can find the α:

$$\alpha = \frac{1}{\sqrt{2T}}\left[\sqrt{\omega\sqrt{\omega^2\mu^2+\gamma^2}-\mu\omega^2}-i\sqrt{\omega\sqrt{\omega^2\mu^2+\gamma^2}+\mu\omega^2}\right] \tag{7}$$

To keep $F(z)$ finite for large z (we're assuming the string extends from $z=0$ to $z=+\infty$), we must have $A=0$ and therefore

$$F(z) = A_T \exp\left\{-\frac{z}{\sqrt{2T}}\left[\sqrt{\omega\sqrt{\omega^2\mu^2+\gamma^2}-\mu\omega^2}-i\sqrt{\omega\sqrt{\omega^2\mu^2+\gamma^2}+\mu\omega^2}\right]\right\} \tag{8}$$

where A_T is the (complex) amplitude of the wave.

We can think of the imaginary part of the exponential as the wave number k_2 and the real part as a spatial decay factor λ. That is

$$k_2 \equiv \frac{1}{\sqrt{2T}}\sqrt{\omega\sqrt{\omega^2\mu^2+\gamma^2}+\mu\omega^2} \tag{9}$$

$$\lambda \equiv \frac{1}{\sqrt{2T}}\sqrt{\omega\sqrt{\omega^2\mu^2+\gamma^2}-\mu\omega^2} \tag{10}$$

$$\alpha = \lambda - ik_2 \tag{11}$$

$$F(z) = A_T e^{-\lambda z + ik_2 z} \tag{12}$$

Because of the negative real part in the exponent, the wave is attenuated (its amplitude falls off with distance) with a characteristic penetration distance d of

$$d = \frac{1}{\lambda} = \frac{\sqrt{2T}}{\sqrt{\omega\sqrt{\omega^2\mu^2+\gamma^2}-\mu\omega^2}} \tag{13}$$

This is the distance at which the amplitude falls to $1/e$ of its value at $z=0$.

Finally, we can consider the case of two strings joined by a massless knot at $z=0$, with the string on the right embedded in the viscous fluid. If the wave is sinusoidal, then we have

$$f(z,t) = \begin{cases} A_I e^{i(k_1 z-\omega t)} + A_R e^{i(-k_1 z-\omega t)} & z<0 \\ e^{-i\omega t}F(z) & z>0 \end{cases} \tag{14}$$

where $F(z)$ is given by 8.

The boundary conditions require the wave function and its derivative to be continuous at $z=0$, so we get

$$A_I + A_R = A_T \tag{15}$$
$$ik_1(A_I - A_R) = -\alpha A_T = -\alpha(A_I + A_R) \tag{16}$$

Solving for A_R we get

$$A_R = -\frac{\alpha + ik_1}{\alpha - ik_1} A_I \tag{17}$$
$$= -\frac{\lambda + i(k_1 - k_2)}{\lambda - i(k_1 + k_2)} A_I \tag{18}$$

If A_I is real (that is, the incident wave has zero phase: $\delta_I = 0$) then the magnitude of the amplitude of the reflected wave is

$$|A_R| = \sqrt{\frac{\lambda^2 + (k_1 - k_2)^2}{\lambda^2 + (k_1 + k_2)^2}} A_I \tag{19}$$

We can convert 18 by multiplying top and bottom by the complex conjugate of the denominator:

$$A_R = -\frac{(\lambda + i(k_1 - k_2))(\lambda + i(k_1 + k_2))}{\lambda^2 + (k_1 + k_2)^2} A_I \tag{20}$$
$$= -\frac{\lambda^2 + k_2^2 - k_1^2 + 2i\lambda k_1}{\lambda^2 + (k_1 + k_2)^2} A_I \tag{21}$$

The phase of the reflected wave is

$$\tan\delta_R = \frac{\Im(A_R)}{\Re(A_R)} \tag{22}$$
$$= \frac{2\lambda k_1}{\lambda^2 + k_2^2 - k_1^2} \tag{23}$$

If $\gamma = 0$ (so there is no fluid surrounding string 2), then $\lambda = 0$ and

$$A_R = -\frac{k_2^2 - k_1^2}{(k_1 + k_2)^2} A_I \tag{24}$$
$$= \frac{k_1 - k_2}{k_2 + k_1} A_I \tag{25}$$
$$\tan\delta_R = 0 \tag{26}$$

which agrees with Griffiths equation 9.29 for sinusoidal waves.