

WAVES IN A VISCOUS FLUID

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.7.

Another example of a transverse wave is that of a string embedded in a viscous fluid which adds a drag force on the string. The drag force is proportional to the string's transverse speed, so we need to insert a term

$$(0.1) \quad \Delta F_{drag} = -\gamma \frac{\partial f}{\partial t} \Delta z$$

where γ is a constant determined by the viscosity of the fluid, into our derivation of the original wave equation. This gives us a modified wave equation:

$$(0.2) \quad T \frac{\partial^2 f}{\partial z^2} - \gamma \frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial t^2}$$

where T is the tension in the string and μ is the mass per unit length of the string.

If we assume that string's frequency ω is constant, then we can write

$$(0.3) \quad f(z, t) = e^{-i\omega t} F(z)$$

and the wave equation reduces to an ODE:

$$(0.4) \quad TF'' + i\omega\gamma F = -\mu\omega^2 F$$

$$(0.5) \quad F'' = -\frac{i\omega\gamma + \mu\omega^2}{T} F \equiv \alpha^2 F$$

The general solution is

$$(0.6) \quad F(z) = Ae^{\alpha z} + Be^{-\alpha z}$$

However, α is a complex number, so F contains both oscillatory and exponential parts.

Using Maple, we can find the α :

$$(0.7) \quad \alpha = \frac{1}{\sqrt{2T}} \left[\sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega^2} - i \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} + \mu \omega^2} \right]$$

To keep $F(z)$ finite for large z (we're assuming the string extends from $z = 0$ to $z = +\infty$), we must have $A = 0$ and therefore

$$(0.8) \quad F(z) = A_T \exp \left\{ -\frac{z}{\sqrt{2T}} \left[\sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega^2} - i \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} + \mu \omega^2} \right] \right\}$$

where A_T is the (complex) amplitude of the wave.

We can think of the imaginary part of the exponential as the wave number k_2 and the real part as a spatial decay factor λ . That is

$$(0.9) \quad k_2 \equiv \frac{1}{\sqrt{2T}} \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} + \mu \omega^2}$$

$$(0.10) \quad \lambda \equiv \frac{1}{\sqrt{2T}} \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega^2}$$

$$(0.11) \quad \alpha = \lambda - ik_2$$

$$(0.12) \quad F(z) = A_T e^{-\lambda z + ik_2 z}$$

Because of the negative real part in the exponent, the wave is attenuated (its amplitude falls off with distance) with a characteristic penetration distance d of

$$(0.13) \quad d = \frac{1}{\lambda} = \frac{\sqrt{2T}}{\sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega^2}}$$

This is the distance at which the amplitude falls to $1/e$ of its value at $z = 0$.

Finally, we can consider the case of two strings joined by a massless knot at $z = 0$, with the string on the right embedded in the viscous fluid. If the wave is sinusoidal, then we have

$$(0.14) \quad f(z, t) = \begin{cases} A_I e^{i(k_1 z - \omega t)} + A_R e^{i(-k_1 z - \omega t)} & z < 0 \\ e^{-i\omega t} F(z) & z > 0 \end{cases}$$

where $F(z)$ is given by 0.8.

The boundary conditions require the wave function and its derivative to be continuous at $z = 0$, so we get

$$(0.15) \quad A_I + A_R = A_T$$

$$(0.16) \quad ik_1(A_I - A_R) = -\alpha A_T = -\alpha(A_I + A_R)$$

Solving for A_R we get

$$(0.17) \quad A_R = -\frac{\alpha + ik_1}{\alpha - ik_1} A_I$$

$$(0.18) \quad = -\frac{\lambda + i(k_1 - k_2)}{\lambda - i(k_1 + k_2)} A_I$$

If A_I is real (that is, the incident wave has zero phase: $\delta_I = 0$) then the magnitude of the amplitude of the reflected wave is

$$(0.19) \quad |A_R| = \sqrt{\frac{\lambda^2 + (k_1 - k_2)^2}{\lambda^2 + (k_1 + k_2)^2}} A_I$$

We can convert 0.18 by multiplying top and bottom by the complex conjugate of the denominator:

$$(0.20) \quad A_R = -\frac{(\lambda + i(k_1 - k_2))(\lambda + i(k_1 + k_2))}{\lambda^2 + (k_1 + k_2)^2} A_I$$

$$(0.21) \quad = -\frac{\lambda^2 + k_2^2 - k_1^2 + 2i\lambda k_1}{\lambda^2 + (k_1 + k_2)^2} A_I$$

The phase of the reflected wave is

$$(0.22) \quad \tan \delta_R = \frac{\Im(A_R)}{\Re(A_R)}$$

$$(0.23) \quad = \frac{2\lambda k_1}{\lambda^2 + k_2^2 - k_1^2}$$

If $\gamma = 0$ (so there is no fluid surrounding string 2), then $\lambda = 0$ and

$$(0.24) \quad A_R = -\frac{k_2^2 - k_1^2}{(k_1 + k_2)^2} A_I$$

$$(0.25) \quad = \frac{k_1 - k_2}{k_2 + k_1} A_I$$

$$(0.26) \quad \tan \delta_R = 0$$

which agrees with Griffiths equation 9.29 for sinusoidal waves.