WAVES: POLARIZATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.8.

For the sinusoidal wave on a string, let's assume that the wave travels in the $+z$ direction and then define x and y axes in the usual way. We can produce the wave by shaking the string in the xz plane, in which case there is no component of the wave in the y direction, or by shaking the string in the yz plane, in which case there is no motion in the x direction. Or we could shake the string in some other plane intermediate between xz and xy, as long as that plane contains the z axis.

If the x axis points upwards and the y axis points horizontally, then a wave moving in the xz plane can be said to be *vertically polarized*, while a wave moving in the yz plane is *horizontally polarized*. (These terms aren't really technical terms, but you get the idea.) Any transverse wave whose motion is restricted to a plane is *linearly polarized*. We can represent this in vector notation by writing

$$f(z,t) = Ae^{i(kz-\omega t)}\hat{\mathbf{n}} \tag{1}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the z axis and parallel to the plane of polarization. Thus for a wave moving in the xz plane, $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ and so on.

If θ is the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{x}}$, then in general

$$\hat{\mathbf{n}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \tag{2}$$

and a wave linearly polarized in the $\hat{\mathbf{n}}$ direction is given by

$$f(z,t) = Ae^{i(kz-\omega t)}(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}) \tag{3}$$

More generally, we can have a wave that is the vector sum of a vertically and horizontally polarized wave:

$$f(z,t) = A_x e^{i(kz-\omega t)}\cos\theta\hat{\mathbf{x}} + A_y e^{i(kz-\omega t)}\sin\theta\hat{\mathbf{y}} \tag{4}$$

where the complex amplitudes are given by

$$A_i = a_i e^{i\delta_i} \tag{5}$$

with $i = x, y$. The real amplitudes a_x and a_y must be the same for both components in order for the polarization to be in the $\hat{\mathbf{n}}$ plane.

For a wave to be linearly polarized, the phases of the two component waves must also be equal, so that $\delta_x = \delta_y = \delta$. In that case, the components of the wave at a fixed location $z = z_0$ vary with time according to

$$\mathbf{f}_x(z_0, t) = a e^{i(kz_0 - \omega t + \delta)} \cos\theta \hat{\mathbf{x}} \tag{6}$$

$$\mathbf{f}_y(z_0, t) = a e^{i(kz_0 - \omega t + \delta)} \sin\theta \hat{\mathbf{y}} \tag{7}$$

Since it is only the real parts of these equations that are physically meaningful, we see that the time dependence is the same in both cases and is $\cos(kz_0 - \omega t + \delta)$. Thus the horizontal and vertical components oscillate in phase, so that the wave remains in the $\hat{\mathbf{n}}$ plane.

Now suppose that $\delta_y = \pi/2$ and $\delta_x = 0$ so that the two components are out of phase. Then the real components of the wave are

$$x = \Re(f_x) = a\cos(kz_0 - \omega t)\cos\theta \tag{8}$$

$$y = \Re(f_y) = a\cos\left(kz_0 - \omega t + \frac{\pi}{2}\right)\sin\theta \tag{9}$$

$$= -a\sin(kz_0 - \omega t)\sin\theta \tag{10}$$

In this case, the point on the string at position $z = z_0$ follows a curve with equation

$$\frac{x^2}{(a\cos\theta)^2} + \frac{y^2}{(a\sin\theta)^2} = 1 \tag{11}$$

which is the equation of an ellipse. In this case, the wave is *elliptically polarized*. If $\theta = \frac{\pi}{4}$ then we get a circle and the wave is *circularly polarized*.

To see which direction the string moves, try a couple of values of t at $z_0 = 0$. For $t = 0$, $x(0,0) = a\cos\theta$, $y(0,0) = 0$. Then for $t = \pi/2\omega$ we have $x(0, \pi/2\omega) = 0$, $y(0, \pi/2\omega) = a\sin\theta$. Thus if θ increases counterclockwise as we look down the z axis towards the origin, the point is rotating counterclockwise. The shape of the string at any given instant of time is a helix (either elliptical or circular, depending on θ). We can generate this by shaking the end of the string in an ellipse or circle.

If we set $\delta_y = -\pi/2$ then

$$x = \Re(f_x) = a\cos(kz_0 - \omega t)\cos\theta \tag{12}$$

$$y = \Re(f_y) = a\cos\left(kz_0 - \omega t - \frac{\pi}{2}\right)\sin\theta \tag{13}$$

$$= a\sin(kz_0 - \omega t)\sin\theta \tag{14}$$

and this time, a point on the string moves clockwise around the z axis.