

## WAVES: POLARIZATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.8.

For the sinusoidal wave on a string, let's assume that the wave travels in the  $+z$  direction and then define  $x$  and  $y$  axes in the usual way. We can produce the wave by shaking the string in the  $xz$  plane, in which case there is no component of the wave in the  $y$  direction, or by shaking the string in the  $yz$  plane, in which case there is no motion in the  $x$  direction. Or we could shake the string in some other plane intermediate between  $xz$  and  $xy$ , as long as that plane contains the  $z$  axis.

If the  $x$  axis points upwards and the  $y$  axis points horizontally, then a wave moving in the  $xz$  plane can be said to be *vertically polarized*, while a wave moving in the  $yz$  plane is *horizontally polarized*. (These terms aren't really technical terms, but you get the idea.) Any transverse wave whose motion is restricted to a plane is *linearly polarized*. We can represent this in vector notation by writing

$$f(z, t) = Ae^{i(kz - \omega t)} \hat{\mathbf{n}} \quad (1)$$

where  $\hat{\mathbf{n}}$  is a unit vector perpendicular to the  $z$  axis and parallel to the plane of polarization. Thus for a wave moving in the  $xz$  plane,  $\hat{\mathbf{n}} = \hat{\mathbf{x}}$  and so on.

If  $\theta$  is the angle between  $\hat{\mathbf{n}}$  and  $\hat{\mathbf{x}}$ , then in general

$$\hat{\mathbf{n}} = \cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}} \quad (2)$$

and a wave linearly polarized in the  $\hat{\mathbf{n}}$  direction is given by

$$f(z, t) = Ae^{i(kz - \omega t)} (\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}) \quad (3)$$

More generally, we can have a wave that is the vector sum of a vertically and horizontally polarized wave:

$$f(z, t) = A_x e^{i(kz - \omega t)} \cos \theta \hat{\mathbf{x}} + A_y e^{i(kz - \omega t)} \sin \theta \hat{\mathbf{y}} \quad (4)$$

where the complex amplitudes are given by

$$A_i = a_i e^{i\delta_i} \quad (5)$$

with  $i = x, y$ . The real amplitudes  $a_x$  and  $a_y$  must be the same for both components in order for the polarization to be in the  $\hat{\mathbf{n}}$  plane.

For a wave to be linearly polarized, the phases of the two component waves must also be equal, so that  $\delta_x = \delta_y = \delta$ . In that case, the components of the wave at a fixed location  $z = z_0$  vary with time according to

$$\mathbf{f}_x(z_0, t) = a e^{i(kz_0 - \omega t + \delta)} \cos \theta \hat{\mathbf{x}} \quad (6)$$

$$\mathbf{f}_y(z_0, t) = a e^{i(kz_0 - \omega t + \delta)} \sin \theta \hat{\mathbf{y}} \quad (7)$$

Since it is only the real parts of these equations that are physically meaningful, we see that the time dependence is the same in both cases and is  $\cos(kz_0 - \omega t + \delta)$ . Thus the horizontal and vertical components oscillate in phase, so that the wave remains in the  $\hat{\mathbf{n}}$  plane.

Now suppose that  $\delta_y = \pi/2$  and  $\delta_x = 0$  so that the two components are out of phase. Then the real components of the wave are

$$x = \Re(f_x) = a \cos(kz_0 - \omega t) \cos \theta \quad (8)$$

$$y = \Re(f_y) = a \cos\left(kz_0 - \omega t + \frac{\pi}{2}\right) \sin \theta \quad (9)$$

$$= -a \sin(kz_0 - \omega t) \sin \theta \quad (10)$$

In this case, the point on the string at position  $z = z_0$  follows a curve with equation

$$\frac{x^2}{(a \cos \theta)^2} + \frac{y^2}{(a \sin \theta)^2} = 1 \quad (11)$$

which is the equation of an ellipse. In this case, the wave is *elliptically polarized*. If  $\theta = \frac{\pi}{4}$  then we get a circle and the wave is *circularly polarized*.

To see which direction the string moves, try a couple of values of  $t$  at  $z_0 = 0$ . For  $t = 0$ ,  $x(0, 0) = a \cos \theta$ ,  $y(0, 0) = 0$ . Then for  $t = \pi/2\omega$  we have  $x(0, \pi/2\omega) = 0$ ,  $y(0, \pi/2\omega) = a \sin \theta$ . Thus if  $\theta$  increases counterclockwise as we look down the  $z$  axis towards the origin, the point is rotating counterclockwise. The shape of the string at any given instant of time is a helix (either elliptical or circular, depending on  $\theta$ ). We can generate this by shaking the end of the string in an ellipse or circle.

If we set  $\delta_y = -\pi/2$  then

$$x = \Re(f_x) = a \cos(kz_0 - \omega t) \cos \theta \quad (12)$$

$$y = \Re(f_y) = a \cos\left(kz_0 - \omega t - \frac{\pi}{2}\right) \sin \theta \quad (13)$$

$$= a \sin(kz_0 - \omega t) \sin \theta \quad (14)$$

and this time, a point on the string moves clockwise around the  $z$  axis.