MAXWELL STRESS TENSOR AND ELECTROMAGNETIC WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.12.

The Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}$ gives the components of momentum flux density. That is, the component $-T_{ij}$ is the momentum component i per unit area per unit time crossing a surface normal to the j direction. The formula is

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2 \right) \tag{1}$$

For a monochromatic plane wave with amplitude A, direction $\hat{\mathbf{z}}$, frequency ω and phase δ polarized in the x direction, we have

$$\mathbf{E} = E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{x}} \tag{2}$$

$$\mathbf{B} = \frac{E_0}{c}\cos(kz - \omega t + \delta)\,\hat{\mathbf{y}} \tag{3}$$

All the off-diagonal elements of $\overleftrightarrow{\mathbf{T}}$ are zero, and the diagonal elements are, using $c = 1/\sqrt{\epsilon_0 \mu_0}$:

$$T_{xx} = \frac{1}{2}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) - \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t + \delta) \tag{4}$$

$$= \frac{1}{2}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) - \frac{1}{2}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \tag{5}$$

$$= 0 \tag{6}$$

$$T_{yy} = -\frac{1}{2}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) + \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t + \delta) \tag{7}$$

$$= 0 \tag{8}$$

$$T_{zz} = -\frac{1}{2}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) - \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t + \delta) \tag{9}$$

$$= -\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \tag{10}$$

Thus there is no momentum flux in the x or y directions, and the momentum flux density in the z direction is $\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$. Since the average of $\cos^2 x$ over a single cycle is $\frac{1}{2}$, the average momentum flux density is

$$\langle -T_{zz} \rangle = \frac{1}{2}\epsilon_0 E_0^2 \tag{11}$$

Since the wave moves with speed c, a unit area sweeps out a volume c in unit time. Thus the average momentum density (that is, momentum per unit volume) is

$$\langle \mathfrak{p} \rangle = \frac{1}{c}\langle -T_{zz} \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \tag{12}$$

which agrees with our earlier result.

Also in the earlier post, we showed that rate per unit area of energy flow is given by the Poynting vector and is

$$\mathbf{S} = \epsilon_0 c E_0^2 \cos^2(kz - \omega t + \delta)\hat{\mathbf{z}} \tag{13}$$

$$= -cT_{zz}\hat{\mathbf{z}} \tag{14}$$

The energy *density* is thus $S/c = -T_{zz}$ so the energy density is the same as the momentum flux density.