

## MAXWELL STRESS TENSOR AND ELECTROMAGNETIC WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.12.

The Maxwell stress tensor  $\overleftrightarrow{\mathbf{T}}$  gives the components of momentum flux density. That is, the component  $-T_{ij}$  is the momentum component  $i$  per unit area per unit time crossing a surface normal to the  $j$  direction. The formula is

$$T_{ij} \equiv \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \quad (1)$$

For a monochromatic plane wave with amplitude  $A$ , direction  $\hat{\mathbf{z}}$ , frequency  $\omega$  and phase  $\delta$  polarized in the  $x$  direction, we have

$$\mathbf{E} = E_0 \cos(kz - \omega t + \delta) \hat{\mathbf{x}} \quad (2)$$

$$\mathbf{B} = \frac{E_0}{c} \cos(kz - \omega t + \delta) \hat{\mathbf{y}} \quad (3)$$

All the off-diagonal elements of  $\overleftrightarrow{\mathbf{T}}$  are zero, and the diagonal elements are, using  $c = 1/\sqrt{\epsilon_0 \mu_0}$ :

$$T_{xx} = \frac{1}{2} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) - \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t + \delta) \quad (4)$$

$$= \frac{1}{2} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) - \frac{1}{2} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \quad (5)$$

$$= 0 \quad (6)$$

$$T_{yy} = -\frac{1}{2} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) + \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t + \delta) \quad (7)$$

$$= 0 \quad (8)$$

$$T_{zz} = -\frac{1}{2} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) - \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(kz - \omega t + \delta) \quad (9)$$

$$= -\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \quad (10)$$

Thus there is no momentum flux in the  $x$  or  $y$  directions, and the momentum flux density in the  $z$  direction is  $\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$ . Since the

average of  $\cos^2 x$  over a single cycle is  $\frac{1}{2}$ , the average momentum flux density is

$$\langle -T_{zz} \rangle = \frac{1}{2} \epsilon_0 E_0^2 \quad (11)$$

Since the wave moves with speed  $c$ , a unit area sweeps out a volume  $c$  in unit time. Thus the average momentum density (that is, momentum per unit volume) is

$$\langle \mathbf{p} \rangle = \frac{1}{c} \langle -T_{zz} \rangle = \frac{1}{2c} \epsilon_0 E_0^2 \quad (12)$$

which agrees with our earlier result.

Also in the earlier post, we showed that rate per unit area of energy flow is given by the Poynting vector and is

$$\mathbf{S} = \epsilon_0 c E_0^2 \cos^2(kz - \omega t + \delta) \hat{\mathbf{z}} \quad (13)$$

$$= -c T_{zz} \hat{\mathbf{z}} \quad (14)$$

The energy *density* is thus  $S/c = -T_{zz}$  so the energy density is the same as the momentum flux density.