THREE LAWS OF GEOMETRICAL OPTICS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.15.

We got the equations for reflected and transmitted waves when an electromagnetic wave is incident on a boundary head-on. Using similar (although somewhat more involved) methods, we can solve the general case where a wave is incident on a boundary at any angle.

Suppose the wave vector of the incident wave is $\mathbf{k}_I$ and the boundary is located on the xy plane. The incident wave is

$$\tilde{\mathbf{E}}_I = \tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} \tag{1}$$

$$\tilde{\mathbf{B}}_I = \frac{1}{v_1}\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I \tag{2}$$

where the direction of polarization is given by the direction of $\tilde{\mathbf{E}}_{0_I}$. At this stage, we don't know the direction of either the reflected or transmitted waves, so we can just write them as

$$\tilde{\mathbf{E}}_R = \tilde{\mathbf{E}}_{0_R} e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} \tag{3}$$

$$\tilde{\mathbf{B}}_R = \frac{1}{v_1}\hat{\mathbf{k}}_R \times \tilde{\mathbf{E}}_R \tag{4}$$

$$\tilde{\mathbf{E}}_T = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)} \tag{5}$$

$$\tilde{\mathbf{B}}_T = \frac{1}{v_2}\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T \tag{6}$$

The speeds v_i of the waves are determined by the medium and not by the direction of the waves and since the frequency ω is the same for all three waves and $\omega = kv$, we get

$$k_I v_1 = k_R v_1 \tag{7}$$

$$k_I = k_R \tag{8}$$

$$k_I v_1 = k_T v_2 \tag{9}$$

$$k_T = \frac{v_1}{v_2}k_I = \frac{n_2}{n_1}k_I \tag{10}$$

where n_i is the index of refraction.

This gives us the magnitudes of the wave vectors, but not the directions. To get those, we need to apply the boundary conditions for zero surface charge:

$$\begin{aligned}
\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} &= 0 &(11)\\
B_1^{\perp} - B_2^{\perp} &= 0 &(12)\\
\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} &= 0 &(13)\\
\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} &= 0 &(14)
\end{aligned}$$

where the subscript 1 refers to fields in medium 1 ($z < 0$) and 2 refers to medium 2 ($z > 0$). That is

$$\begin{aligned}
\mathbf{E}_1 &= \mathbf{E}_I + \mathbf{E}_R &(15)\\
\mathbf{E}_2 &= \mathbf{E}_T &(16)
\end{aligned}$$

and similarly for $\mathbf{B}$. Thus all these boundary conditions have the form

$$A_I e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + A_R e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} = A_T e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)} \tag{17}$$

for coefficients A_i that vary, depending on the boundary condition being imposed, but that don't depend on $\mathbf{r}$ or t. The space and time dependencies are entirely within the complex exponentials. Since we've taken the boundary to be the xy plane, $z = 0$ in these equations, so they become

$$\begin{aligned}
A_I e^{i\left(k_{Ix}x+k_{Iy}y-\omega t\right)} + A_R e^{i\left(k_{Rx}x+k_{Ry}y-\omega t\right)} &= A_T e^{i\left(k_{Tx}x+k_{Ty}y-\omega t\right)} &(18)\\
A_I e^{i\left(k_{Ix}x+k_{Iy}y\right)} + A_R e^{i\left(k_{Rx}x+k_{Ry}y\right)} &= A_T e^{i\left(k_{Tx}x+k_{Ty}y\right)} &(19)
\end{aligned}$$

These equations must be valid over the entire xy plane, so they are valid for $x = 0$ or $y = 0$:

$$A_I e^{ik_{Ix}x} + A_R e^{ik_{Rx}x} = A_T e^{ik_{Tx}x} \tag{20}$$

To see what this implies, consider the general case

$$Ae^{iax} + Be^{ibx} = Ce^{icx} \tag{21}$$

for all x. That is, both sides are the same function of x, so they must have the same Taylor expansion. Starting at point x and expanding to get the function at $x+\Delta x$ we get up to first order in Δx:

$$Ae^{iax}\left(1+ia\Delta x\right)+Be^{ibx}\left(1+ib\Delta x\right)+\mathcal{O}\left(\Delta x^{2}\right)=Ce^{icx}\left(1+ic\Delta x\right)+\mathcal{O}\left(\Delta x^{2}\right) \tag{22}$$

Using 21 and cancelling common factors we get

$$\begin{aligned} aAe^{iax}+bBe^{ibx} &= cCe^{icx} && (23)\\ &= c\left(Ae^{iax}+Be^{ibx}\right) && (24)\\ A\left(a-c\right)e^{iax}+B\left(b-c\right)e^{ibx} &= 0 && (25) \end{aligned}$$

This last equation must be true for all x so we must have

$$a=b=c \tag{26}$$

Going back to 19, we therefore must have

$$\begin{aligned} k_{Ix} &= k_{Rx}=k_{Tx} && (27)\\ k_{Iy} &= k_{Ry}=k_{Ty} && (28) \end{aligned}$$

If we choose axes so that $\mathbf{k}_I$ is in the xz plane, then $\mathbf{k}_R$ and $\mathbf{k}_T$ must also lie in the same plane. This gives the first law of geometrical optics:

The wave vectors of the incident, reflected and transmitted waves all lie in the same plane, and this plane also contains the normal to the boundary.

If the angle of incidence is θ_I (that is, the angle between $\mathbf{k}_I$ and the normal to the xy plane) with θ_R and θ_T the corresponding angles for reflection and transmission, then from 8 we have

$$\begin{aligned} k_I\sin\theta_I &= k_R\sin\theta_R && (29)\\ &= k_I\sin\theta_R && (30)\\ \sin\theta_I &= \sin\theta_R && (31)\\ \theta_I &= \theta_R && (32) \end{aligned}$$

This is the familiar condition for specular reflection: the angle of incidence is equal to the angle of reflection. This is the second law of geometrical optics.

Finally, for the transmitted wave, from 10 we have

$$k_I \sin\theta_I = k_T \sin\theta_T \quad (33)$$
$$= \frac{n_2}{n_1} k_I \sin\theta_T \quad (34)$$
$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \quad (35)$$

This is Snell's law, or the law of refraction, which is the third law of geometrical optics.

We haven't actually applied the boundary conditions yet, apart from 20, but we'll look at that next to get an idea of the reflection and transmission coefficients.

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