SKIN DEPTH OF ELECTROMAGNETIC WAVES IN CONDUCTORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.18b-c.

We've seen that any free charge within a conductor migrates to the surface with a characteristic time that depends on the conductor's conductance and permittivity. Once that transient effect subsides, we can take the free charge density to be zero: $\rho_f = 0$. This doesn't mean that the free current density $\mathbf{J}_f$ is zero, though. We can still have a current in an electrically neutral conductor caused by electrons moving relative to stationary atomic nuclei.

In linear media, Maxwell's equations are

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon}\rho_f \tag{1}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{2}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{B} = \mu\mathbf{J}_f + \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

Within a conductor with conductivity σ (*not* surface charge density!) Ohm's law can be written as

$$\mathbf{J}_f = \sigma\mathbf{E} \tag{5}$$

so with $\rho_f = 0$, Maxwell's equations become

$$\nabla\cdot\mathbf{E} = 0 \tag{6}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{7}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{8}$$

$$\nabla\times\mathbf{B} = \mu\sigma\mathbf{E} + \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t} \tag{9}$$

We can take the curl of the last two equations in the same way as when we derived the wave equation in a vacuum.

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E} \tag{10}$$

$$= -\nabla\times\frac{\partial\mathbf{B}}{\partial t} \tag{11}$$

$$= -\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}-\mu\sigma\frac{\partial\mathbf{E}}{\partial t} \tag{12}$$

Since $\nabla\cdot\mathbf{E}=0$ we get

$$\nabla^2\mathbf{E}=\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}+\mu\sigma\frac{\partial\mathbf{E}}{\partial t} \tag{13}$$

A similar calculation for **B** gives

$$\nabla^2\mathbf{B}=\mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2}+\mu\sigma\frac{\partial\mathbf{B}}{\partial t} \tag{14}$$

We thus get the wave equation modified by the addition of an extra first-derivative term. Conveniently, these equations have a similar solution to the ordinary wave equation. For a plane wave travelling in the z direction

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)} \tag{15}$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)} \tag{16}$$

Substituting 15 into 13 we get, after cancelling terms

$$\tilde{k}^2=\omega^2\mu\varepsilon+i\omega\mu\sigma \tag{17}$$

The fact that the wave vector $\tilde{k}$ is complex means that the resulting wave has both an oscillatory and an exponentially decaying factor. Finding the square root of this using Maple gives

(18)
$$\tilde{k}=\frac{1}{2}\sqrt{2\sqrt{\omega^4\mu^2\varepsilon^2+\omega^2\mu^2\sigma^2}+2\,\omega^2\mu\,\varepsilon}+\frac{1}{2}i\sqrt{2\sqrt{\omega^4\mu^2\varepsilon^2+\omega^2\mu^2\sigma^2}-2\,\omega^2\mu\,\varepsilon}$$
(19)
$$=\frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1}+i\frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1}$$
(20)
$$\equiv k+i\kappa$$

The wave is then given by

(21) $$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z}e^{i(kz-\omega t)}$$

(22) $$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z}e^{i(kz-\omega t)}$$

If an electromagnetic wave hits a conductor (starting in air or vacuum, say), the wave will attenuate as it penetrates the conductor with a characteristic distance, called the *skin depth* d of

(23) $$d=\frac{1}{\kappa}=\frac{1}{\omega}\left[\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)\right]^{-1/2}$$

The skin depth depends not only on the conductivity and permittivity, but also on the frequency of the incident radiation.

Example 1. The skin depth for good conductors such as metals is very small for a wide range of frequencies. For silver, the conductivity is $\sigma = 6.30\times10^7\ \mathrm{S\ m^{-1}}$. To get the permittivity of silver, we can return to its definition:

(24) $$\varepsilon=\varepsilon_0(1+\chi_e)$$

where the electric susceptibility χ_e is defined by the amount of polarization in the material that is produced by an applied electric field:

(25) $$\mathbf{P}=\varepsilon_0\chi_e\mathbf{E}$$

For a perfect conductor, no polarization is produced by any amount of electric field, so $\chi_e=0$ and $\varepsilon=\varepsilon_0$. Thus it's a reasonable approximation for a good conductor like silver to take $\varepsilon \approx \varepsilon_0$. Most materials also have a permeability $\mu \approx \mu_0$. At a microwave frequency of 10^{10} Hz $= 2\pi \times 10^{10}$ s^{-1}

$$\frac{\sigma}{\varepsilon_0 \omega} = 1.13 \times 10^8 \tag{26}$$

so to a good approximation

$$d = \frac{1}{\omega}\sqrt{\frac{2\omega}{\mu_0 \sigma}} = \sqrt{\frac{2}{\omega \mu_0 \sigma}} = 6.34 \times 10^{-7}\text{m} \tag{27}$$

The real part of $\tilde{k}$ gives the oscillatory part of the wave, so the wavelength is

$$\lambda = \frac{2\pi}{k} \tag{28}$$

and the wave speed is

$$v = \nu\lambda = \frac{\omega}{2\pi}\lambda = \frac{\omega}{k} \tag{29}$$

The index of refraction is the ratio of c to v as usual:

$$n = \frac{c}{v} = \frac{ck}{\omega} \tag{30}$$

Example 2. For copper, $\sigma = 5.96 \times 10^7$ S m^{-1}. For radio waves with a frequency of 1 MHz $= 2\pi \times 10^6$ s^{-1} we have

$$\begin{aligned}
\frac{\sigma}{\varepsilon_0 \omega} &= 1.07 \times 10^{12} && (31)\\
k &\approx \sqrt{\frac{\omega \mu_0 \sigma}{2}} && (32)\\
&= 1.533 \times 10^4 \text{ m}^{-1} && (33)\\
\lambda &= \frac{2\pi}{k} = 4.1 \times 10^{-4}\text{m} && (34)
\end{aligned}$$

The wave speed is

$$v = \frac{\omega}{k} = 410 \text{ m s}^{-1} \tag{35}$$

This is obviously very slow for electromagnetic radiation.

In air or vacuum, $v = c = 3 \times 10^8 \mathrm{m\ s}^{-1}$ and $\lambda = v/\nu = 300$ m which is closer to what we'd expect for radio waves.

PINGBACKS

Pingback: Skin depth of water and metals
Pingback: Electromagnetic waves in conductors: phases and amplitudes
Pingback: Reflection at a conducting surface: the physics of mirrors