ELECTROMAGNETIC WAVES IN CONDUCTORS: PHASES AND AMPLITUDES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.19c.

Electromagnetic waves in a conductor(where there is free current but no free charge) can be written as

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i\left(\tilde{k}z-\omega t\right)} \tag{1}$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i\left(\tilde{k}z-\omega t\right)} \tag{2}$$

where the wave vector is complex:

$$\tilde{k} = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1}+i\frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1} \equiv k+i\kappa \tag{3}$$

so

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{4}$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{5}$$

By applying Maxwell's equations in a conductorwe can get a few more properties of these waves. The equations are

$$\nabla\cdot\mathbf{E} = 0 \tag{6}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{7}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{8}$$

$$\nabla\times\mathbf{B} = \mu\sigma\mathbf{E}+\mu\epsilon\frac{\partial\mathbf{E}}{\partial t} \tag{9}$$

Using the same techniques as in analyzing waves in vacuum. Both $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{B} = 0$ from which we get

$$\begin{aligned}
\nabla\cdot\tilde{\mathbf{E}} &= (ik-\kappa)\,\tilde{E}_{0z}e^{i(kz-\omega t)}=0 && (10)\\
\nabla\cdot\tilde{\mathbf{B}} &= (ik-\kappa)\,\tilde{B}_{0z}e^{i(kz-\omega t)}=0 && (11)
\end{aligned}$$

Since this must be true for all z, we must have

$$\tilde{E}_{0z}=\tilde{B}_{0z}=0 \tag{12}$$

That is, the wave has only x and y components, so it must be a transverse wave: a wave that oscillates in a plane perpendicular to the direction of propagation. If we orient the axes so that **E** is polarized in the x direction then

$$\tilde{\mathbf{E}}(z,t)=\tilde{E}_0e^{-\kappa z}e^{i(kz-\omega t)}\hat{\mathbf{x}} \tag{13}$$

Applying 8 to this gives

$$\begin{aligned}
\nabla\times\tilde{\mathbf{E}} &= i(k+i\kappa)\,\tilde{E}_0e^{-\kappa z}e^{i(kz-\omega t)}\hat{\mathbf{y}} && (14)\\
&= i\tilde{k}\tilde{E}_0e^{-\kappa z}e^{i(kz-\omega t)}\hat{\mathbf{y}} && (15)\\
&= -\frac{\partial\mathbf{B}}{\partial t} && (16)\\
&= i\omega\tilde{\mathbf{B}}_0e^{-\kappa z}e^{i(kz-\omega t)} && (17)\\
\tilde{\mathbf{B}}(z,t) &= \frac{\tilde{k}}{\omega}\tilde{E}_0e^{-\kappa z}e^{i(kz-\omega t)}\hat{\mathbf{y}} && (18)
\end{aligned}$$

As in vacuum, **E** and **B** are perpendicular and transverse to the direction of propagation. Unlike in the vacuum, however, the two components of the wave may not be in phase, due to the presence of the complex variable $\tilde{k}$ in the equation for $\tilde{\mathbf{B}}$. If we write $\tilde{k}$ in modulus-phase form we have

$$\tilde{k}=Ke^{i\phi} \tag{19}$$

where

$$\begin{aligned}
K &= \sqrt{k^2+\kappa^2} && (20)\\
&= \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} && (21)\\
\phi &= \arctan\frac{\kappa}{k} && (22)
\end{aligned}$$

Then the complex amplitudes of the two components can be written as

$$\tilde{E}_0 = E_0 e^{i\delta_E} \tag{23}$$

$$\tilde{B}_0 = \frac{K}{\omega} E_0 e^{i(\delta_E+\phi)} \tag{24}$$

and the ratio of the real amplitudes is

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \tag{25}$$

Example. For a good conductor, $\sigma \gg \epsilon\omega$ so from 3 $k \approx \kappa$ so from 22 the phase difference between **B** and **E** is $\pi/4$. The ratio of amplitudes is

$$\frac{B_0}{E_0} = \sqrt{\frac{\sigma\mu}{\omega}} \tag{26}$$

For a typical good conductor $\sigma \approx 10^7$S m^{-1} and $\mu \approx \mu_0$ and at visible frequencies $\omega \approx 10^{15}$ s^{-1} so

$$\frac{B_0}{E_0} = 1.12 \times 10^{-7} \text{ s m}^{-1} \tag{27}$$

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