ELECTROMAGNETIC WAVES IN CONDUCTORS: ENERGY DENSITY AND INTENSITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.20.

We can write the electromagnetic wave inside a conductor as (if we orient the axes so that $\mathbf{E}$ is polarized in the x direction)

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{\mathbf{x}} \tag{0.1}$$

$$= E_0 e^{-\kappa z} e^{i(kz-\omega t+\delta_E)}\hat{\mathbf{x}} \tag{0.2}$$

$$\tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{\mathbf{y}} \tag{0.3}$$

$$= \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} E_0 e^{-\kappa z} e^{i(kz-\omega t+\delta_E+\phi)}\hat{\mathbf{y}} \tag{0.4}$$

where

$$\tilde{k} = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1} + i\frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1} \equiv k+i\kappa \equiv Ke^{i\phi} \tag{0.5}$$

The actual fields are the real parts of these equations, so

$$\mathbf{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{\mathbf{x}} \tag{0.6}$$

$$\mathbf{B}(z,t) = \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\hat{\mathbf{y}} \tag{0.7}$$

The energy density in the wave is

$$u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right) \tag{0.8}$$

Taking the time average (over one cycle) of this we have (since the average of $\cos^2\omega t$ over one cycle $\tau = 2\pi/\omega$ is $\frac{1}{2}$):

$$u=\frac{E_0^2e^{-2\kappa z}}{4}\left(\varepsilon+\varepsilon\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right) \tag{0.9}$$

For a good conductor, $\sigma\gg\varepsilon\omega$ so

$$u \approx \frac{E_0^2e^{-2\kappa z}}{4}\left(\varepsilon+\frac{\sigma}{\omega}\right) \tag{0.10}$$

$$\approx \frac{E_0^2e^{-2\kappa z}}{4}\frac{\sigma}{\omega} \tag{0.11}$$

From 0.10, we see that the magnetic contribution (σ/ω) is much larger than the electric contribution (ε) for a good conductor.

We can express this in terms of the wave vector k by using 0.5 for a good conductor.

$$k \approx \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} \tag{0.12}$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}} \tag{0.13}$$

$$\sigma = \frac{2k^2}{\omega\mu} \tag{0.14}$$

$$u \approx \frac{E_0^2e^{-2\kappa z}}{2}\frac{k^2}{\mu\omega^2} \tag{0.15}$$

The intensity is the energy crossing a unit area in unit time, which is the energy density times the volume crossing a unit area per unit time, which is

$$I=uv \tag{0.16}$$

where v is the speed of the wave, which is ω/k so

$$I=\frac{E_0^2e^{-2\kappa z}}{2}\frac{k}{\mu\omega} \tag{0.17}$$