

ELECTROMAGNETIC WAVES IN CONDUCTORS: ENERGY DENSITY AND INTENSITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.20.

We can write the electromagnetic wave inside a conductor as (if we orient the axes so that \mathbf{E} is polarized in the x direction)

$$(1) \quad \tilde{\mathbf{E}}(z, t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} \hat{\mathbf{x}}$$

$$(2) \quad = E_0 e^{-\kappa z} e^{i(kz - \omega t + \delta_E)} \hat{\mathbf{x}}$$

$$(3) \quad \tilde{\mathbf{B}}(z, t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} \hat{\mathbf{y}}$$

$$(4) \quad = \sqrt{\epsilon\mu \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}} E_0 e^{-\kappa z} e^{i(kz - \omega t + \delta_E + \phi)} \hat{\mathbf{y}}$$

where

$$(5) \quad \tilde{k} = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1} + i \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1} \equiv k + i\kappa \equiv Ke^{i\phi}$$

The actual fields are the real parts of these equations, so

$$(6) \quad \mathbf{E}(z, t) = E_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E) \hat{\mathbf{x}}$$

$$(7) \quad \mathbf{B}(z, t) = \sqrt{\epsilon\mu \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}} E_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi) \hat{\mathbf{y}}$$

The energy density in the wave is

$$(8) \quad u = \frac{1}{2} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right)$$

Taking the time average (over one cycle) of this we have (since the average of $\cos^2 \omega t$ over one cycle $\tau = 2\pi/\omega$ is $\frac{1}{2}$):

$$(9) \quad u = \frac{E_0^2 e^{-2\kappa z}}{4} \left(\epsilon + \epsilon \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2} \right)$$

For a good conductor, $\sigma \gg \epsilon \omega$ so

$$(10) \quad u \approx \frac{E_0^2 e^{-2\kappa z}}{4} \left(\epsilon + \frac{\sigma}{\omega} \right)$$

$$(11) \quad \approx \frac{E_0^2 e^{-2\kappa z}}{4} \frac{\sigma}{\omega}$$

From 10, we see that the magnetic contribution (σ/ω) is much larger than the electric contribution (ϵ) for a good conductor.

We can express this in terms of the wave vector k by using 5 for a good conductor.

$$(12) \quad k \approx \frac{\omega \sqrt{\mu \epsilon}}{\sqrt{2}} \sqrt{\frac{\sigma}{\epsilon \omega}}$$

$$(13) \quad = \sqrt{\frac{\omega \mu \sigma}{2}}$$

$$(14) \quad \sigma = \frac{2k^2}{\omega \mu}$$

$$(15) \quad u \approx \frac{E_0^2 e^{-2\kappa z}}{2} \frac{k^2}{\mu \omega^2}$$

The intensity is the energy crossing a unit area in unit time, which is the energy density times the volume crossing a unit area per unit time, which is

$$(16) \quad I = uv$$

where v is the speed of the wave, which is ω/k so

$$(17) \quad I = \frac{E_0^2 e^{-2\kappa z}}{2} \frac{k}{\mu \omega}$$