# REFLECTION AT A CONDUCTING SURFACE: THE PHYSICS OF MIRRORS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.21.

We can analyze reflection of an electromagnetic wave at a nonconductor-conductor interface in a similar way to that used for a nonconductor-nonconductor interface. We'll look only at the case of normal incidence here.

As before, we start with the boundary conditions in linear media derived from Maxwell's equations:

$$\begin{aligned}
\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} &= \sigma_f &(1)\\
B_1^{\perp} - B_2^{\perp} &= 0 &(2)\\
\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} &= 0 &(3)\\
\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} &= \mathbf{K}_f \times \hat{\mathbf{n}} &(4)
\end{aligned}$$

We'll take medium 1 as the nonconductor (air, say) and medium 2 as the conductor. We're allowing for the presence of free surface charge density $\sigma_f$ and free current density $\mathbf{K}_f$ at the boundary.

If we're dealing with a conductor that obey's Ohm's law, the volume current density is proportional to the electric field

$$\mathbf{J}_f = \sigma \mathbf{E} \tag{5}$$

where here $\sigma$ is the conductivity, not a charge density. Recall that $\mathbf{J}_f$ is the amount of current flowing through a unit area in the conductor. If we had a surface current density $\mathbf{K}_f$, this current flows along the boundary as a sheet of moving charge with infinitesimal thickness, so that the cross-sectional area occupied by $\mathbf{K}_f$ is essentially zero, making the *volume* charge density infinite. For a finite conductivity $\sigma$ it would take an infinite electric field to produce this surface current, so we can safely assume that $\mathbf{K}_f = 0$ in what follows.

The incident and reflected waves are both in medium 1, so if we polarize the wave in the $x$ direction, we have for the incident wave:

$$\begin{aligned}\tilde{\mathbf{E}}_I &= \tilde{E}_{0_I} e^{i(k_1 z-\omega t)}\hat{\mathbf{x}} &(6)\\ \tilde{\mathbf{B}}_I &= \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z-\omega t)}\hat{\mathbf{y}} &(7)\end{aligned}$$

where $v_1$ is the speed of the wave in medium 1.

The reflected wave is travelling in the $-z$ direction and has equations

$$\begin{aligned}\tilde{\mathbf{E}}_R &= \tilde{E}_{0_R} e^{i(-k_1 z-\omega t)}\hat{\mathbf{x}} &(8)\\ \tilde{\mathbf{B}}_R &= -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z-\omega t)}\hat{\mathbf{y}} &(9)\end{aligned}$$

The transmitted wave is inside the conductor, so its equations can be written as

$$\begin{aligned}\tilde{\mathbf{E}}_T(z,t) &= \tilde{E}_{0_T} e^{i\left(\tilde{k}_2 z-\omega t\right)}\hat{\mathbf{x}} &(10)\\ \tilde{\mathbf{B}}_T(z,t) &= \frac{\tilde{k}_2}{\omega}\tilde{E}_{0_T} e^{i\left(\tilde{k}_2 z-\omega t\right)}\hat{\mathbf{y}} &(11)\end{aligned}$$

where the wave vector is complex:

$$\tilde{k} = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1}+i\frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1} \equiv k+i\kappa \tag{12}$$

We can now apply the boundary conditions. Equation 1 tells us that $\sigma_f = 0$ since there is no perpendicular component of $\mathbf{E}$ (remember the wave is transverse). Equation 2 tells us nothing ($0 = 0$). From 3, assuming that the boundary is at $z = 0$, we get, since all components of $\mathbf{E}$ are in the $x$ direction:

$$\tilde{E}_{0_I}+\tilde{E}_{0_R} = \tilde{E}_{0_T} \tag{13}$$

Finally, from 4 we get, since all components of $\mathbf{B}$ are in the $y$ direction and $\mathbf{K}_f = 0$:

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0_I}-\tilde{E}_{0_R}\right) = \frac{\tilde{k}_2}{\mu_2\omega}\tilde{E}_{0_T} \tag{14}$$

which we can rewrite as

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2 \tilde{E}_{0_T} \tag{15}$$

$$\equiv \tilde{\beta} \tilde{E}_{0_T} \tag{16}$$

Adding 13 and 16 we get

$$\tilde{E}_{0_I} = \frac{1}{2}\left(1+\tilde{\beta}\right)\tilde{E}_{0_T} \tag{17}$$

$$\tilde{E}_{0_T} = \frac{2}{1+\tilde{\beta}}\tilde{E}_{0_I} \tag{18}$$

Subtracting 13 and 16 we get

$$\tilde{E}_{0_R} = \frac{1}{2}\left(1-\tilde{\beta}\right)\tilde{E}_{0_T} \tag{19}$$

$$= \frac{1-\tilde{\beta}}{1+\tilde{\beta}}\tilde{E}_{0_I} \tag{20}$$

These are deceptively simple equations, since everything with a tilde on it is a complex number. To get the actual amplitudes and phases we need to extract the real and imaginary parts.

**Example.** To put some numbers into these equations, let's consider an air-silver interface. For a good conductor such as silver, $\sigma \gg \epsilon\omega$ and in 12

$$k \approx \kappa \approx \sqrt{\frac{\mu\sigma\omega}{2}} \tag{21}$$

In air, $v_1 \approx c$ and we can take $\mu_1 = \mu_2 = \mu_0$ so from 16

$$\tilde{\beta} \approx c\sqrt{\frac{\mu_0\sigma}{2\omega}}\,(1+i) \tag{22}$$

For silver $\sigma = 6\times 10^7$ S m$^{-1}$ and at an optical frequency of $\omega = 4\times 10^{15}$ s$^{-1}$ we get

$$\tilde{\beta} = 29.1\,(1+i) \tag{23}$$

$$\frac{1-\tilde{\beta}}{1+\tilde{\beta}} = \frac{-28.1-29.1i}{30.1+29.1i}\times\frac{30.1-29.1i}{30.1-29.1i} \tag{24}$$

$$= \frac{1}{1753}\,(-1693-58.2i) \tag{25}$$

To get the reflection coefficient we can write the complex amplitudes in modulus-phase form as

$$\tilde{E}_{0_R} = E_{0_R} e^{i\delta_R} \quad (26)$$

$$\tilde{E}_{0_I} = E_{0_I} e^{i\delta_I} \quad (27)$$

The intensity of a wave is the average over one cycle of the magnitude of the Poynting vector, so the fact that the incident and reflected waves may have different phases doesn't matter (since they have the same frequency). This means that

$$R = \frac{\frac{1}{2}c\epsilon_0 E_{0_R}^2}{\frac{1}{2}c\epsilon_0 E_{0_I}^2} \quad (28)$$

$$\frac{|E_{0_R}|^2}{|E_{0_I}|^2} \quad (29)$$

$$= \frac{1693^2 + 58.2^2}{1753^2} \quad (30)$$

$$= 0.93 \quad (31)$$

Silver reflects 93% of the incident light, so it makes a good mirror.