

## FREQUENCY DEPENDENCE OF ELECTRIC PERMITTIVITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.23a.

Experimentally, it is known that the permittivity of a material when an electromagnetic wave passes through it depends on the frequency of the wave. To develop a (relatively crude) theory of how this comes about, it's worth recalling the definition of permittivity  $\epsilon$ , which arises from the ability of an external electric field  $\mathbf{E}$  to polarize a dielectric, producing a polarization density  $\mathbf{P}$ :

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad (1)$$

$\chi_e$  is the electric susceptibility, and the permittivity is defined in terms of it by

$$\epsilon = \epsilon_0 (1 + \chi_e) \quad (2)$$

Therefore, if we want to discover the dependence of  $\epsilon$  on frequency, we might start by trying to find a relation between  $\mathbf{P}$  and  $\mathbf{E}$ , where  $\mathbf{E}$  arises from the electromagnetic wave passing through the material. The idea is to look at a typical electron bound to one of the atoms in the dielectric and work out the dipole moment of this atom in terms of the applied field in the wave.

The electron (with charge  $q$ ) is subject to several forces. First, there is the force from the wave. The wave's electric component has the form (assuming it's polarized in the  $x$  direction):

$$\tilde{\mathbf{E}} = \tilde{E}_0 e^{i(kz - \omega t)} \hat{\mathbf{x}} \quad (3)$$

For a fixed point, say  $z = 0$ , the field oscillates in place so the force on the electron is the real part of the field times the charge:

$$\mathbf{F}_E = qE_0 \cos \omega t \hat{\mathbf{x}} \quad (4)$$

Second, the electron experiences a binding force with the nucleus. A simple model that we used earlier took the electron to be a sphere of uniform charge density, of radius  $a$  (the Bohr radius in hydrogen, which is  $5.29 \times 10^{-11}$  m) centred on the nucleus. In this case, when the electron is displaced a distance  $x$  from equilibrium, the binding force is

$$\mathbf{F}_b = -\frac{Zq^2}{4\pi\epsilon_0 a^3} x \hat{\mathbf{x}} \quad (5)$$

where  $Z$  is the atomic number (number of protons in the nucleus) and the minus sign is because  $\mathbf{F}_b$  pulls the electron back towards equilibrium. This force is a harmonic oscillator force, since

$$\mathbf{F}_b = -k_b x \hat{\mathbf{x}} \quad (6)$$

$$k_b \equiv \frac{Zq^2}{4\pi\epsilon_0 a^3} \quad (7)$$

The harmonic oscillator force has a natural frequency of

$$\omega_0 = \sqrt{\frac{k_b}{m}} \quad (8)$$

so we can write the binding force as

$$\mathbf{F}_b = -m\omega_0^2 x \hat{\mathbf{x}} \quad (9)$$

**Example 1.** We can work out this natural frequency for a hydrogen-like atom with  $Z = 1$ . We get

$$k_b = \frac{(1.6 \times 10^{-19})^2}{4\pi(8.85 \times 10^{-12})(5.29 \times 10^{-11})^3} \quad (10)$$

$$= 1.55 \times 10^3 \text{ kg s}^{-2} \quad (11)$$

$$\omega_0 = \sqrt{\frac{k_b}{m}} \quad (12)$$

$$= \sqrt{\frac{1.55 \times 10^3}{9.1 \times 10^{-31}}} \quad (13)$$

$$= 4.13 \times 10^{16} \text{ s}^{-1} \quad (14)$$

$$\nu_0 = \frac{\omega_0}{2\pi} = 6.58 \times 10^{15} \text{ s}^{-1} \quad (15)$$

This frequency is in the near ultraviolet, just beyond the violet end of the visible spectrum.

Finally, there will be a damping force because, once the wave is turned off, we expect the electron to eventually return to its equilibrium position. This can happen by radiating away energy or from interactions with other

fields in the material. The simplest damping force is proportional to, and opposite in direction to, the velocity, so we can let

$$\mathbf{F}_d = -\gamma m \dot{\mathbf{x}} \quad (16)$$

where  $\gamma$  is the damping constant.

Since all the forces act in the  $x$  direction, we can drop the vector notation and apply  $F_{total} = m\ddot{x}$  to get

$$m\ddot{x} = -\gamma m \dot{x} - m\omega_0^2 x + qE_0 \cos \omega t \quad (17)$$

At this point, we can do the usual trick of introducing complex numbers by defining  $x$  to be the real part of a complex variable  $\tilde{x}$ . This equation then becomes

$$\ddot{\tilde{x}} + \gamma \dot{\tilde{x}} + \omega_0^2 \tilde{x} = \frac{q}{m} E_0 e^{-i\omega t} \quad (18)$$

The solution is

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t} \quad (19)$$

since if we substitute this into the ODE we get

$$-\omega^2 \tilde{x}_0 e^{-i\omega t} - i\omega \gamma \tilde{x}_0 e^{-i\omega t} + \omega_0^2 \tilde{x}_0 e^{-i\omega t} = \frac{q}{m} E_0 e^{-i\omega t} \quad (20)$$

$$\tilde{x}_0 = \frac{qE_0}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \quad (21)$$

The dipole moment  $p$  of the atom is the charge times the separation, which is given by  $x$ , so it is the real part of

$$\tilde{p} = q\tilde{x}_0 \quad (22)$$

$$= \frac{q^2 E_0}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} e^{-i\omega t} \quad (23)$$

$$\tilde{\mathbf{p}} = \frac{q^2}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \tilde{\mathbf{E}} \quad (24)$$

We have now achieved our objective (for a single electron) since we have the dipole moment in terms of the applied electric field. However, because of the damping term, the *real* part of  $\tilde{p}$  is not directly proportional to  $E = E_0 \cos \omega t$  so the medium isn't linear. We still need to calculate the polarization density  $\mathbf{P}$  to get the permittivity. If there are  $f_j$  electrons with

natural frequency  $\omega_0 = \omega_j$  and damping constant  $\gamma = \gamma_j$  per atom (or molecule) and  $N$  atoms per unit volume, then

$$\tilde{\mathbf{P}} = N \sum_j f_j \tilde{\mathbf{p}}_j \quad (25)$$

$$= \frac{Nq^2}{m} \tilde{\mathbf{E}} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)} \quad (26)$$

Generalizing 1 so that the susceptibility is complex, we have

$$\begin{aligned} \tilde{\mathbf{P}} &= \epsilon_0 \tilde{\chi}_e \tilde{\mathbf{E}} \\ \tilde{\chi}_e &= \frac{Nq^2}{\epsilon_0 m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)} \end{aligned}$$

This gives a complex dielectric constant and permittivity:

$$\tilde{\epsilon}_r = 1 + \tilde{\chi}_e \quad (27)$$

$$= 1 + \frac{Nq^2}{\epsilon_0 m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)} \quad (28)$$

$$\tilde{\epsilon} = \epsilon_0 \tilde{\epsilon}_r \quad (29)$$

$$= \epsilon_0 + \frac{Nq^2}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)} \quad (30)$$

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