

## REFRACTION AND DISPERSION COEFFICIENTS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.23b.

In a dispersive medium, the permittivity depends on the frequency of electromagnetic radiation.

$$\tilde{\epsilon} = \epsilon_0 + \frac{Nq^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad (1)$$

where there are  $f_j$  electrons per atom with natural frequency  $\omega_j$  and damping factor  $\gamma_j$ , and there are  $N$  atoms per unit volume. Because  $\tilde{\epsilon}$  is complex, the medium isn't linear in the sense that the polarization is directly proportional to the applied field, but if we take both the polarization  $\tilde{\mathbf{P}}$  and field  $\tilde{\mathbf{E}}$  to be complex, then the medium is linear in the sense that

$$\tilde{\mathbf{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\mathbf{E}} \quad (2)$$

With this assumption, we can substitute the complex permittivity  $\tilde{\epsilon}$  for the ordinary real permittivity  $\epsilon$  in Maxwell's equations and follow through the same steps to get the wave equation, which now becomes

$$\nabla^2 \tilde{\mathbf{E}} = \mu \tilde{\epsilon} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} \quad (3)$$

Just as before, we can get plane wave solutions of the form

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)} \quad (4)$$

where  $\tilde{k}$  is a complex wave vector

$$\tilde{k} = \sqrt{\tilde{\epsilon} \mu} \omega \quad (5)$$

The actual real and imaginary parts of  $\tilde{k}$  are complicated expressions since  $\tilde{\epsilon}$  is a sum of complex numbers, but we can use the shortcut notation

$$\tilde{k} = k + i\kappa \quad (6)$$

giving (assuming  $\mathbf{E}$  is polarized in the  $x$  direction):

$$\tilde{\mathbf{E}}(z, t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} \hat{\mathbf{x}} \quad (7)$$

The intensity of the radiation is proportional to  $|\tilde{\mathbf{E}}|^2$  so the intensity falls off according to  $e^{-2\kappa z}$  as we penetrate the medium. The *absorption coefficient* is defined as

$$\alpha \equiv 2\kappa \quad (8)$$

and gives a measure of the reciprocal of the distance at which the intensity is attenuated.

We can write the complex permittivity in 1 as

$$\tilde{\epsilon} = \epsilon_0 + \frac{Nq^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \left[ \frac{\omega_j^2 - \omega^2 + i\omega\gamma_j}{\omega_j^2 - \omega^2 + i\omega\gamma_j} \right] \quad (9)$$

$$= \epsilon_0 + \frac{Nq^2}{m} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} + i \frac{Nq^2}{m} \sum_j \frac{f_j \omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} \quad (10)$$

If we stay away from the resonant frequencies, where  $\omega \approx \omega_j$ , the sum terms are quite small so we can approximate them in 5 by the first order term in a Taylor expansion. If we also take  $\mu \approx \mu_0$  as is true of most materials, and use  $c = 1/\sqrt{\mu_0\epsilon_0}$ , we get

$$\tilde{k} = \sqrt{\frac{\tilde{\epsilon}}{\epsilon_0 c^2}} \omega \quad (11)$$

Using  $\sqrt{1+x} \approx 1 + \frac{1}{2}x$  for small  $x$ , we get

$$\tilde{k} \approx \frac{\omega}{c} \left[ 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} \right] + i \frac{\omega}{c} \left[ \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j \omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} \right] \quad (12)$$

From 7 the speed of the wave is

$$v = \frac{\omega}{k} \quad (13)$$

so the index of refraction is

$$n = \frac{c}{v} \quad (14)$$

$$\approx 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} \quad (15)$$

and the absorption coefficient is

$$\alpha = 2\kappa \quad (16)$$

$$\approx \frac{Nq^2\omega^2}{c\epsilon_0 m} \sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + (\omega\gamma_j)^2} \quad (17)$$

If we stay away from resonances, the damping term becomes insignificant so the index of refraction is approximately

$$n \approx 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2} \quad (18)$$

If the frequency  $\omega$  of the wave is significantly less than all the resonant frequencies  $\omega_j$  we can further approximate this using  $\frac{1}{1-x} \approx 1+x$  for small  $x$ :

$$n \approx 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 (1 - \omega^2/\omega_j^2)} \quad (19)$$

$$\approx 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2} + \omega^2 \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^4} \quad (20)$$

In a vacuum,  $c = \lambda v = \lambda \omega / 2\pi$  so

$$n \approx 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2} + \frac{4\pi^2 c^2}{\lambda^2} \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^4} \quad (21)$$

$$= 1 + A \left( 1 + \frac{B}{\lambda^2} \right) \quad (22)$$

where

$$A = \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2} \quad (23)$$

$$B = \frac{4\pi^2 c^2}{A} \frac{Nq^2}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^4} \quad (24)$$

Eqn 22 is known as the Cauchy formula, although Cauchy had many equations named after him (particularly in the area of complex variable theory), so the name is easily confused with other formulas. The parameter  $A$  is the *coefficient of refraction* and  $B$  is the *coefficient of dispersion*. The more usual form of Cauchy's equation seems to be  $n = 1 + A + \frac{B}{\lambda^2}$  (I couldn't find any sources that gave the formula in the form used by Griffiths).

**Example.** Applying this model to hydrogen at 0 C and atmospheric pressure (that is, standard temperature and pressure, or STP), the number of electrons per molecule of  $H_2$  is  $f_j = 2$ . In the previous post, we found that the resonant frequency is  $\omega_0 = 4.13 \times 10^{16} \text{ s}^{-1}$ . At STP, an ideal gas occupies  $22.414 \text{ m}^3 \text{ kmol}^{-1}$ , so the number density is

$$N = \frac{(6.02 \times 10^{23})(1000)}{22.414} = 2.69 \times 10^{25} \quad (25)$$

The parameters are

$$A = \frac{Nq^2}{2\epsilon_0 m} \frac{2}{\omega_0^2} = 5 \times 10^{-5} \quad (26)$$

$$B = \frac{4\pi^2 c^2}{A} \frac{Nq^2}{2\epsilon_0 m} \frac{2}{\omega_0^4} = \frac{4\pi^2 c^2}{\omega_0^2} = 2.08 \times 10^{-15} \text{ m}^2 \quad (27)$$

The experimental values quoted by Griffiths are

$$A = 1.36 \times 10^{-4} \quad (28)$$

$$B = 7.7 \times 10^{-15} \quad (29)$$

so the values from the model are at least around the right order of magnitude.

#### PINGBACKS

Pingback: Resonances in a dispersive medium

Pingback: Group velocity of electromagnetic waves in a dispersive medium