RESONANCES IN A DISPERSIVE MEDIUM

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.24.

In a dispersive medium, the index of refraction is

$$n \approx 1+\frac{Nq^2}{2\epsilon_0 m}\sum_j \frac{f_j\left(\omega_j^2-\omega^2\right)}{\left(\omega_j^2-\omega^2\right)^2+\left(\omega\gamma_j\right)^2} \tag{0.1}$$

and the absorption coefficient is

$$\alpha \approx \frac{Nq^2\omega^2}{c\epsilon_0 m}\sum_j \frac{f_j\gamma_j}{\left(\omega_j^2-\omega^2\right)^2+\left(\omega\gamma_j\right)^2} \tag{0.2}$$

We can look at the behaviour of these coefficients near one of the resonances, that is, when $\omega \approx \omega_j$ for some j. To simplify things, we'll assume that there is only one term in the sum (that is, only one resonance). In practice, when we're near one resonance, the other resonances don't affect things much unless they are very close together.

In this case, we'll take the one natural frequency to be ω_0 and the associated damping coefficient to be γ_0, and define

$$x \equiv \frac{\omega}{\omega_0} \tag{0.3}$$

Then we get, after dividing top and bottom by ω_0^2:

$$n \approx 1+\frac{Nq^2 f_0}{2\epsilon_0 m}\frac{1-x^2}{\omega_0^2\left(1-x^2\right)^2+\gamma_0^2x^2} \tag{0.4}$$

$$\alpha \approx \frac{Nq^2 f_0\gamma_0}{m\epsilon_0 c}\frac{x^2}{\omega_0^2\left(1-x^2\right)^2+\gamma_0^2x^2} \tag{0.5}$$

The index of refraction n rises to a peak when ω is just before ω_0, then dips sharply, reaching a minimum just after ω_0, after which it rises slowly again. We can find the maximum and minimum by setting the derivative to zero and solving for x. Using Maple to simplify the result, we get

$$\frac{dn}{dx}=\frac{Nq^2f_0}{\varepsilon_0 m}\frac{x\left(x^4-2x^2+1-\frac{\gamma_0^2}{\omega_0^2}\right)}{\omega_0^2\left(x^4\left(\frac{\gamma_0^2}{\omega_0^2}-2\right)x^2+1\right)}=0 \tag{0.6}$$

The roots are

$$x=0,\pm\sqrt{1+\frac{\gamma_0}{\omega_0}},\pm\sqrt{1-\frac{\gamma_0}{\omega_0}} \tag{0.7}$$

The negative and zero roots aren't of interest, so we see that n reaches its maximum at $x=\sqrt{1-\frac{\gamma_0}{\omega_0}}$ and minimum at $x=\sqrt{1+\frac{\gamma_0}{\omega_0}}$. If $\gamma_0\ll\omega_0$, these approximate to

$$\omega_{n\,max} \approx \omega_0-\frac{1}{2}\gamma_0 \tag{0.8}$$

$$\omega_{n\,min} \approx \omega_0+\frac{1}{2}\gamma_0 \tag{0.9}$$

so the width of the anomalous region is γ_0.

From 0.5, we see that the absorption reaches a maximum when $x=1$ (this can be checked by calculating the derivative) and has a value of

$$\alpha_{max}=\frac{Nq^2f_0}{m\varepsilon_0 c\gamma_0} \tag{0.10}$$

Substituting the positive roots from 0.7 into 0.5 and dividing by α_{max} we find

$$\frac{\alpha_{n\,max}}{\alpha_{max}} = \frac{\omega_0-\gamma_0}{2\omega_0-\gamma_0}\approx\frac{1}{2} \tag{0.11}$$

$$\frac{\alpha_{n\,min}}{\alpha_{max}} = \frac{\omega_0+\gamma_0}{2\omega_0+\gamma_0}\approx\frac{1}{2} \tag{0.12}$$

Thus for small γ, the index of refraction reaches its maximum and minimum values roughly where the absorption is half its maximum.

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