GROUP VELOCITY OF ELECTROMAGNETIC WAVES IN A DISPERSIVE MEDIUM

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.25.

In a dispersive medium, the permittivity is a complex quantity given by

$$\tilde{\varepsilon}=\varepsilon_0+\frac{Nq^2}{m}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j} \tag{1}$$

and the wave vector is also complex:

$$\tilde{k}=\sqrt{\tilde{\varepsilon}\mu}\,\omega \tag{2}$$

If the sum term in 1 is small compared to ε_0, we can approximate it in the square root using $\sqrt{1+x}\approx 1+\frac{1}{2}x$.

$$\begin{aligned}
\tilde{k} &= \omega\sqrt{\mu\varepsilon_0}\left[1+\frac{Nq^2}{m\varepsilon_0}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}\right]^{1/2} && (3)\\
&\approx \frac{\omega}{c}\left(1+\frac{Nq^2}{2m\varepsilon_0}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}\right) && (4)
\end{aligned}$$

where we've taken $\mu\approx\mu_0$ and used $\sqrt{\mu_0\varepsilon_0}=\frac{1}{c}$.

The solution to the wave equation is

$$\tilde{\mathbf{E}}(z,t)=\tilde{\mathbf{E}}_0 e^{i\left(\tilde{k}z-\omega t\right)} \tag{5}$$

If the damping factors γ_j are all small and we're away from any resonance frequencies (ω_j), then we can ignore the imaginary term in the denominator and get a real wave vector:

$$k=\frac{\omega}{c}\left(1+\frac{Nq^2}{2m\varepsilon_0}\sum_j\frac{f_j}{\omega_j^2-\omega^2}\right) \tag{6}$$

We can get the group velocity $\frac{d\omega}{dk}$ of a wave packet by implicit differentiation:

$$1=\frac{d\omega}{dk}\left[\frac{1}{c}+\frac{Nq^{2}}{2m\varepsilon_{0}c}\left(\sum_{j}\frac{f_{j}}{\omega_{j}^{2}-\omega^{2}}+\omega\sum_{j}\frac{2f_{j}\omega}{\left(\omega_{j}^{2}-\omega^{2}\right)^{2}}\right)\right] \tag{7}$$

$$\frac{d\omega}{dk}=c\left[1+\frac{Nq^{2}}{2m\varepsilon_{0}}\sum_{j}\frac{f_{j}}{\omega_{j}^{2}-\omega^{2}}\left(1+\frac{2\omega^{2}}{\omega_{j}^{2}-\omega^{2}}\right)\right]^{-1} \tag{8}$$

$$=c\left[1+\frac{Nq^{2}}{2m\varepsilon_{0}}\sum_{j}\frac{f_{j}}{\omega_{j}^{2}-\omega^{2}}\left(\frac{\omega_{j}^{2}-\omega^{2}}{\omega_{j}^{2}-\omega^{2}}+\frac{2\omega^{2}}{\omega_{j}^{2}-\omega^{2}}\right)\right]^{-1} \tag{9}$$

$$=c\left[1+\frac{Nq^{2}}{2m\varepsilon_{0}}\sum_{j}\frac{f_{j}\left(\omega_{j}^{2}+\omega^{2}\right)}{\left(\omega_{j}^{2}-\omega^{2}\right)^{2}}\right]^{-1} \tag{10}$$

Since everything inside the square brackets is positive, we see that $\frac{d\omega}{dk}<c$ for all frequencies. On the other hand, from 6 we see that the phase velocity is

$$\frac{\omega}{k}=c\left(1+\frac{Nq^{2}}{2m\varepsilon_{0}}\sum_{j}\frac{f_{j}}{\omega_{j}^{2}-\omega^{2}}\right)^{-1} \tag{11}$$

which can actually exceed c for some frequencies near the resonant frequencies, since the denominator in the sum can be negative there. Griffiths states that ordinarily, energy (and thus information) carried by the wave travels at the group velocity which is always less that c. However, it *is* possible for the group velocity to exceed c in some cases.

PINGBACKS

Pingback: Wave guide: energy flows at the group velocity