WAVE GUIDES: DERIVATION OF THE WAVE EQUATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.26.

A wave guide is a hollow tube which allows electromagnetic waves to travel down it. Wave guides are usually made of conductors, so we'll assume that they are made of a perfect conductor so that $\mathbf{E}=0$ everywhere inside the conducting boundary.

The boundary conditions implied by Maxwell's equations are

$$\begin{aligned}
\epsilon_1 E_1^{\perp}-\epsilon_2 E_2^{\perp} &= \sigma_f &(1)\\
B_1^{\perp}-B_2^{\perp} &= 0 &(2)\\
\mathbf{E}_1^{\parallel}-\mathbf{E}_2^{\parallel} &= 0 &(3)\\
\frac{1}{\mu_1}\mathbf{B}_1^{\parallel}-\frac{1}{\mu_2}\mathbf{B}_2^{\parallel} &= \mathbf{K}_f\times\hat{\mathbf{n}} &(4)
\end{aligned}$$

In particular, 3 tells us that the parallel component of $\mathbf{E}$ is zero at the boundary of the wave guide. As usual, we'll take the z axis to be parallel to wave guide's axis, so the waves have the form

$$\begin{aligned}
\tilde{\mathbf{E}} &= \tilde{\mathbf{E}}_0(x,y)\,e^{i(kz-\omega t)} &(5)\\
\tilde{\mathbf{B}} &= \tilde{\mathbf{B}}_0(x,y)\,e^{i(kz-\omega t)} &(6)
\end{aligned}$$

Maxwell's equations inside the guide (assumed to be a vacuum) are

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 0 &(7)\\
\nabla\cdot\mathbf{B} &= 0 &(8)\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} &(9)\\
\nabla\times\mathbf{B} &= \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &(10)
\end{aligned}$$

Previously, we applied the divergence equations to show that the waves were transverse (no z component), but that relied on the waves being unbounded plane waves, which is not the case here. It turns out that waves in a wave guide are not transverse in general, in that at least one of **E** and **B** must have a longitudinal component. We therefore write

$$\begin{aligned}
\tilde{\mathbf{E}}_0(x,y) &= E_x\hat{\mathbf{x}}+E_y\hat{\mathbf{y}}+E_z\hat{\mathbf{z}} &(11)\\
\tilde{\mathbf{B}}_0(x,y) &= B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}} &(12)
\end{aligned}$$

where all the components on the RHS depend on x and y, and may be complex functions. Putting these together with 5 and 6 into 9, we get (remembering that the components do not depend on z):

$$\begin{aligned}
\nabla\times\mathbf{E} &= (\partial_y E_z - ikE_y)\hat{\mathbf{x}}+(-\partial_x E_z+ikE_x)\hat{\mathbf{y}}+(\partial_x E_y-\partial_y E_x)\hat{\mathbf{z}} &(13)\\
&= -\frac{\partial\mathbf{B}}{\partial t} &(14)\\
&= i\omega B_x\hat{\mathbf{x}}+i\omega B_y\hat{\mathbf{y}}+i\omega B_z\hat{\mathbf{z}} &(15)
\end{aligned}$$

Equating components, we get

$$\begin{aligned}
\partial_y E_z - ikE_y &= i\omega B_x &(16)\\
-\partial_x E_z + ikE_x &= i\omega B_y &(17)\\
\partial_x E_y - \partial_y E_x &= i\omega B_z &(18)
\end{aligned}$$

We can apply exactly the same procedure to 10 to get the analogous equations

$$\begin{aligned}
\partial_y B_z - ikB_y &= -i\frac{\omega}{c^2}E_x &(19)\\
-\partial_x B_z + ikB_x &= -i\frac{\omega}{c^2}E_y &(20)\\
\partial_x B_y - \partial_y B_x &= -i\frac{\omega}{c^2}E_z &(21)
\end{aligned}$$

We can solve these 6 equations to get the x and y components in terms of the z components. For example, multiplying 17 through by k and 19 through by ω and adding, we get

$$\begin{aligned}
-k\partial_x E_z + ik^2E_x - i\frac{\omega^2}{c^2}E_x &= \omega\partial_y B_z &(22)\\
E_x &= \frac{1}{i(k^2-\omega^2/c^2)}(k\partial_x E_z+\omega\partial_y B_z) &(23)\\
&= \frac{i}{\omega^2/c^2-k^2}(k\partial_x E_z+\omega\partial_y B_z) &(24)
\end{aligned}$$

Similarly, we can get the other 3 equations. Multiply 16 by k and 20 by ω and subtract to get

$$E_y = \frac{i}{\omega^2/c^2 - k^2}\left(k\partial_y E_z - \omega\partial_x B_z\right) \tag{25}$$

Multiply 16 by ω/c^2 and 20 by k and subtract to get

$$B_x = \frac{i}{\omega^2/c^2 - k^2}\left(k\partial_x B_z - \frac{\omega}{c^2}\partial_y E_z\right) \tag{26}$$

Multiply 17 by ω/c^2 and 19 by k and add to get

$$B_y = \frac{i}{\omega^2/c^2 - k^2}\left(k\partial_y B_z + \frac{\omega}{c^2}\partial_x E_z\right) \tag{27}$$

To get the wave equations we can apply 7 and 8:

$$\nabla\cdot\mathbf{E} = \frac{i}{\omega^2/c^2 - k^2}\left[\left(k\partial_{xx}E_z + \omega\partial_{yx}B_z\right) + \left(k\partial_{yy}E_z - \omega\partial_{xy}B_z\right)\right] + ikE_z \tag{28}$$

$$= \frac{i}{\omega^2/c^2 - k^2}\left[k\partial_{xx}E_z + k\partial_{yy}E_z\right] + ikE_z \tag{29}$$

$$= 0 \tag{30}$$

The wave equation for E_z is thus

$$\left[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2\right]E_z = 0 \tag{31}$$

Exactly the same procedure applied to $\nabla\cdot\mathbf{B} = 0$ gives

$$\left[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2\right]B_z = 0 \tag{32}$$

Solving these two equations subject to the boundary conditions will give us all 3 components of each field.

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