RECTANGULAR WAVE GUIDES: TRANSVERSE ELECTRIC WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.27.

The wave equations for an electromagnetic wave in a wave guide are

$$\left[\partial_{xx}+\partial_{yy}+\omega^{2}/c^{2}-k^{2}\right]E_{z} = 0 \tag{1}$$

$$\left[\partial_{xx}+\partial_{yy}+\omega^{2}/c^{2}-k^{2}\right]B_{z} = 0 \tag{2}$$

If $E_z=0$ we have a *transverse electric* or TE wave, so we need to solve only the B_z equation. For a rectangular wave guide with dimensions of a in the x direction and b in the y direction, we can use separation of variables to get a solution (the technique is mathematically the same as that used in solving the infinite square well in quantum mechanics). That is, we assume that

$$B_{z}(x,y)=X(x)Y(y) \tag{3}$$

and substitute this into 2, then divide through by XY:

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{\omega^{2}}{c^{2}}-k^{2}=0 \tag{4}$$

Because this equation must hold for all values of x and y, the two derivative terms must separately be constant, so we have

$$X'' = -k_{x}^{2}X \tag{5}$$

$$Y'' = -k_{y}^{2}Y \tag{6}$$

for some constants k_x and k_y, which satisfy

$$-k_{x}^{2}-k_{y}^{2}+\frac{\omega^{2}}{c^{2}}-k^{2}=0 \tag{7}$$

The general solution to these ODEs can be written as either trigonometric functions or complex exponentials. If we choose trig functions, we have

$$X(x) = A\sin k_x x + B\cos k_x x \tag{8}$$

$$Y(y) = C\sin k_y y + D\cos k_y y \tag{9}$$

The boundary conditions require that at $x=0$ and $x=a$

$$B_1^{\perp} - B_2^{\perp} = 0 \tag{10}$$

The component of $\mathbf{B}$ perpendicular to the walls of the guide in the x direction is B_x, and since $\mathbf{B}=0$ inside the conducting wall, this condition requires $B_x = 0$. In our derivation of the wave equation for a wave guide, we found that

$$B_x = \frac{i}{\omega^2/c^2 - k^2}\left(k\partial_x B_z - \frac{\omega}{c^2}\partial_y E_z\right) \tag{11}$$

and since $E_z = 0$, we must also have $\partial_x B_z = 0$, which in turn requires that

$$X'(0) = X'(a) = 0 \tag{12}$$

From 8, this means that $A=0$ and

$$-k_x B\sin k_x a = 0 \tag{13}$$

so

$$k_x = \frac{m\pi}{a} \tag{14}$$

for m equal to a non-negative integer.

Exactly the same analysis on Y gives us

$$k_y = \frac{n\pi}{b} \tag{15}$$

so the separation of variables solution gives us

$$B_z(x,y) = B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{16}$$

This is known as the TE_{mn} mode. From 7 we get the wave number

$$k=\sqrt{\frac{\omega^2}{c^2}-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \tag{17}$$

In fact, at least one of m or n must be non-zero, as we can see by the following argument. Suppose $m=n=0$. Then $k=\frac{\omega}{c}$. We need the results we got in the previous post:

$$\begin{aligned}
\partial_y E_z - ikE_y &= i\omega B_x && (18)\\
-\partial_x E_z + ikE_x &= i\omega B_y && (19)\\
\partial_x E_y - \partial_y E_x &= i\omega B_z && (20)
\end{aligned}$$

$$\begin{aligned}
\partial_y B_z - ikB_y &= -i\frac{\omega}{c^2}E_x && (21)\\
-\partial_x B_z + ikB_x &= -i\frac{\omega}{c^2}E_y && (22)\\
\partial_x B_y - \partial_y B_x &= -i\frac{\omega}{c^2}E_z && (23)
\end{aligned}$$

With $k=\frac{\omega}{c}$ and $E_z=0$ we get from 21 and 19

$$\begin{aligned}
\partial_y B_z - i\frac{\omega}{c}B_y &= -i\frac{\omega}{c^2}E_x && (24)\\
i\frac{\omega}{c^2}E_x &= i\frac{\omega}{c}B_y && (25)
\end{aligned}$$

Adding these equations gives us

$$\partial_y B_z = 0 \tag{26}$$

Similarly, from 22 and 18 we get

$$\partial_x B_z = 0 \tag{27}$$

Therefore, B_z is constant in both the x and y directions, so it is a constant overall. To find what constant this is, we can use Faraday's law in integral form:

$$\oint \mathbf{E}\cdot d\ell = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} \tag{28}$$

The area of integration on the RHS that we'll choose is a cross-section of the wave guide, so the path of integration on the LHS is around the rectangular boundary of the guide. We know that

$$\mathbf{B} = \mathbf{B}_0 e^{i(kz-\omega t)} \tag{29}$$

so

$$\frac{\partial \mathbf{B}}{\partial t} = -i\omega \mathbf{B}_0 e^{i(kz-\omega t)} \tag{30}$$

In this case $d\mathbf{a}$ points in the z direction, so we get

$$\oint \mathbf{E}\cdot d\ell = i\omega e^{i(kz-\omega t)} \int B_z da \tag{31}$$

$$= i\omega ab e^{i(kz-\omega t)} B_z \tag{32}$$

since B_z is constant over the area of integration.

Since we've chosen the path of integration to be the boundary of the wave guide, we can use the boundary condition

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \tag{33}$$

The field inside the conducting wall of the wave guide is zero, so the parallel field at the boundary must also be zero and, since $\mathbf{E}\cdot d\ell = E^{\parallel} d\ell$, the line integral must also be zero, so we must have

$$B_z = 0 \tag{34}$$

That is, if $m = n = 0$, both the electric and magnetic fields must be transverse (known as TEM mode). Griffiths shows in section 9.5 of his book that in this case, for a hollow wave guide of any cross section, the electric field must actually be zero everywhere, which means no wave can propagate down the guide. Thus the TE_{00} mode cannot exist in a hollow wave guide.

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