

RECTANGULAR WAVE GUIDES: TRANSVERSE ELECTRIC WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.27.

The wave equations for an electromagnetic wave in a wave guide are

$$(0.1) \quad [\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] E_z = 0$$

$$(0.2) \quad [\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] B_z = 0$$

If $E_z = 0$ we have a *transverse electric* or TE wave, so we need to solve only the B_z equation. For a rectangular wave guide with dimensions of a in the x direction and b in the y direction, we can use separation of variables to get a solution (the technique is mathematically the same as that used in solving the infinite square well in quantum mechanics). That is, we assume that

$$(0.3) \quad B_z(x,y) = X(x)Y(y)$$

and substitute this into 0.2, then divide through by XY :

$$(0.4) \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{\omega^2}{c^2} - k^2 = 0$$

Because this equation must hold for all values of x and y , the two derivative terms must separately be constant, so we have

$$(0.5) \quad X'' = -k_x^2 X$$

$$(0.6) \quad Y'' = -k_y^2 Y$$

for some constants k_x and k_y , which satisfy

$$(0.7) \quad -k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0$$

The general solution to these ODEs can be written as either trigonometric functions or complex exponentials. If we choose trig functions, we have

$$(0.8) \quad X(x) = A \sin k_x x + B \cos k_x x$$

$$(0.9) \quad Y(y) = C \sin k_y y + D \cos k_y y$$

The boundary conditions require that at $x = 0$ and $x = a$

$$(0.10) \quad B_1^\perp - B_2^\perp = 0$$

The component of \mathbf{B} perpendicular to the walls of the guide in the x direction is B_x , and since $\mathbf{B} = 0$ inside the conducting wall, this condition requires $B_x = 0$. In our derivation of the wave equation for a wave guide, we found that

$$(0.11) \quad B_x = \frac{i}{\omega^2/c^2 - k^2} \left(k \partial_x B_z - \frac{\omega}{c^2} \partial_y E_z \right)$$

and since $E_z = 0$, we must also have $\partial_x B_z = 0$, which in turn requires that

$$(0.12) \quad X'(0) = X'(a) = 0$$

From 0.8, this means that $A = 0$ and

$$(0.13) \quad -k_x B \sin k_x a = 0$$

so

$$(0.14) \quad k_x = \frac{m\pi}{a}$$

for m equal to a non-negative integer.

Exactly the same analysis on Y gives us

$$(0.15) \quad k_y = \frac{n\pi}{b}$$

so the separation of variables solution gives us

$$(0.16) \quad B_z(x, y) = B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

This is known as the TE_{mn} mode. From 0.7 we get the wave number

$$(0.17) \quad k = \sqrt{\frac{\omega^2}{c^2} - \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}$$

In fact, at least one of m or n must be non-zero, as we can see by the following argument. Suppose $m = n = 0$. Then $k = \frac{\omega}{c}$. We need the results we got in the previous post:

$$(0.18) \quad \partial_y E_z - ikE_y = i\omega B_x$$

$$(0.19) \quad -\partial_x E_z + ikE_x = i\omega B_y$$

$$(0.20) \quad \partial_x E_y - \partial_y E_x = i\omega B_z$$

$$(0.21) \quad \partial_y B_z - ikB_y = -i\frac{\omega}{c^2} E_x$$

$$(0.22) \quad -\partial_x B_z + ikB_x = -i\frac{\omega}{c^2} E_y$$

$$(0.23) \quad \partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2} E_z$$

With $k = \frac{\omega}{c}$ and $E_z = 0$ we get from 0.21 and 0.19

$$(0.24) \quad \partial_y B_z - i\frac{\omega}{c} B_y = -i\frac{\omega}{c^2} E_x$$

$$(0.25) \quad i\frac{\omega}{c^2} E_x = i\frac{\omega}{c} B_y$$

Adding these equations gives us

$$(0.26) \quad \partial_y B_z = 0$$

Similarly, from 0.22 and 0.18 we get

$$(0.27) \quad \partial_x B_z = 0$$

Therefore, B_z is constant in both the x and y directions, so it is a constant overall. To find what constant this is, we can use Faraday's law in integral form:

$$(0.28) \quad \oint \mathbf{E} \cdot d\ell = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

The area of integration on the RHS that we'll choose is a cross-section of the wave guide, so the path of integration on the LHS is around the rectangular boundary of the guide. We know that

$$(0.29) \quad \mathbf{B} = \mathbf{B}_0 e^{i(kz - \omega t)}$$

so

$$(0.30) \quad \frac{\partial \mathbf{B}}{\partial t} = -i\omega \mathbf{B}_0 e^{i(kz - \omega t)}$$

In this case $d\mathbf{a}$ points in the z direction, so we get

$$(0.31) \quad \oint \mathbf{E} \cdot d\ell = i\omega e^{i(kz - \omega t)} \int B_z da$$

$$(0.32) \quad = i\omega ab e^{i(kz - \omega t)} B_z$$

since B_z is constant over the area of integration.

Since we've chosen the path of integration to be the boundary of the wave guide, we can use the boundary condition

$$(0.33) \quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

The field inside the conducting wall of the wave guide is zero, so the parallel field at the boundary must also be zero and, since $\mathbf{E} \cdot d\ell = E^{\parallel} d\ell$, the line integral must also be zero, so we must have

$$(0.34) \quad B_z = 0$$

That is, if $m = n = 0$, both the electric and magnetic fields must be transverse (known as TEM mode). Griffiths shows in section 9.5 of his book that in this case, for a hollow wave guide of any cross section, the electric field must actually be zero everywhere, which means no wave can propagate down the guide. Thus the TE_{00} mode cannot exist in a hollow wave guide.

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