RECTANGULAR WAVE GUIDES: TE MODES AND THE CUTOFF FREQUENCY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.28.

The possible TE modes in a rectangular wave guide are given by

$$B_z(x,y) = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{1}$$

where m and n are integers and a and b are the dimensions of the rectangle, with $a \geq b$. These values are related to the wave number by

$$k = \sqrt{\frac{\omega^2}{c^2} - \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \tag{2}$$

so any modes where

$$\omega^2 < c^2\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right) \tag{3}$$

are not allowed, as they would give an imaginary value of k. The frequency

$$\omega_{mn} = \pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \tag{4}$$

is therefore the cutoff frequency for the mode TE$_{mn}$.

As an example, suppose we had a wave guide with $a = 2.28$ cm, $b = 1.01$ cm and a frequency of $\omega = 1.70\times10^{10}$ Hz. Then the condition for a real k is

$$
\begin{aligned}
\frac{\omega^2}{\pi^2 c^2} &\geq \frac{m^2}{a^2}+\frac{n^2}{b^2} && (5)\\
\frac{4\nu^2}{c^2} &\geq \frac{m^2}{a^2}+\frac{n^2}{b^2} && (6)\\
1.284\times 10^4 &\geq 10^4\left(\frac{m^2}{5.2}+\frac{n^2}{1.02}\right) && (7)\\
1.02m^2+5.2n^2 &\leq 6.813 && (8)
\end{aligned}
$$

Since at least one of m and n must be non-zero, the allowed modes are TE_{10}, TE_{20}, TE_{01} and TE_{11} (where the first subscript is m and the second is n).

If we wish to allow only one mode, this would have to be TE_{10} (since that gives the lowest value on the LHS). The angular frequency range that would do this is between ω_{10} and ω_{20}, that is

$$\omega_{10}=\frac{\pi c}{a}\leq\omega<\frac{2\pi c}{a}=\omega_{20} \tag{9}$$

For the values above

$$
\begin{aligned}
\omega_{10} &= 4.134\times 10^{10}\ \mathrm{s}^{-1} && (10)\\
\omega_{20} &= 8.268\times 10^{10}\ \mathrm{s}^{-1} && (11)
\end{aligned}
$$

The corresponding frequencies and wavelengths are

$$
\begin{aligned}
\nu_{10} &= \frac{\omega_{10}}{2\pi}=6.58\times 10^9\ \mathrm{Hz} && (12)\\
v_{20} &= 1.32\times 10^{10}\ \mathrm{Hz} && (13)\\
\lambda_{10} &= \frac{c}{\nu_{10}}=0.0456\ \mathrm{m} && (14)\\
\lambda_{20} &= 0.0228\ \mathrm{m} && (15)
\end{aligned}
$$