WAVE GUIDE: ENERGY FLOWS AT THE GROUP VELOCITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.29.

It's possible for the phase velocity of an electromagnetic wave to exceed c in some situations. In the case of a TE wave in a rectangular wave guide it's possible to show that the energy in the wave actually moves at the group velocity rather than the phase velocity, and that the group velocity is always less than c. The Poynting vector gives the rate per unit time and per unit area at which energy flows, and its time average is given by

$$\langle\mathbf{S}\rangle=\frac{1}{2\mu_0}\Re\left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\right) \tag{1}$$

The surface integral of $\langle\mathbf{S}\rangle$ over the cross section of the wave guide thus gives the energy flux along the guide. Since the normal $d\mathbf{a}$ to the cross section is along the z direction, we need only the z component of $\langle\mathbf{S}\rangle$:

$$\langle S_z\rangle=\frac{1}{2\mu_0}\Re\left(E_xB_y^*-E_yB_x^*\right) \tag{2}$$

We'll also need the time average of the energy density

$$\langle u_{em}\rangle=\frac{1}{4\mu_0}\left(\frac{1}{c^2}\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^*+\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \tag{3}$$

The z component of magnetic field for a TE mode is

$$B_z(x,y)=B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{4}$$

where m and n are integers and a and b are the dimensions of the rectangle, with $a\ge b$. These values are related to the wave number by

$$k=\frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2} \tag{5}$$

where

$$\omega_{mn}=\pi c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \tag{6}$$

is the cutoff frequency for the mode TE_{mn}.

The components of **E** and **B** in the TE mode can be obtained from, where we've set $E_z = 0$:

$$E_x = \frac{i}{\omega^2/c^2 - k^2}\omega\partial_y B_z \tag{7}$$

$$E_y = -\frac{i}{\omega^2/c^2 - k^2}\omega\partial_x B_z \tag{8}$$

$$B_x = \frac{i}{\omega^2/c^2 - k^2}k\partial_x B_z \tag{9}$$

$$B_y = \frac{i}{\omega^2/c^2 - k^2}k\partial_y B_z \tag{10}$$

Substituting 4 into these equations, we get

$$E_x = -i\frac{\pi n\omega B_0}{\alpha b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{11}$$

$$E_y = i\frac{\pi m\omega B_0}{\alpha a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{12}$$

$$B_x = -i\frac{\pi mkB_0}{\alpha a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{13}$$

$$B_y = -i\frac{\pi nkB_0}{\alpha b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{14}$$

where

$$\alpha \equiv \frac{\omega^2}{c^2} - k^2 \tag{15}$$

We now get

$$\langle S_z\rangle = \frac{\pi^2\omega kB_0^2}{2\alpha^2\mu_0}\left(\frac{n^2}{b^2}\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + \frac{m^2}{a^2}\sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\right) \tag{16}$$

$$\langle u_{em}\rangle = \frac{\pi^2 B_0^2}{\alpha^2\mu_0}\left(\frac{n^2}{b^2}\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + \frac{m^2}{a^2}\sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\right)\left(\frac{\omega^2}{c^2} + k^2\right) \tag{17}$$

$$+\frac{B_0^2}{4\mu_0}\cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}$$

We can now integrate these two quantities over the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$. This is straightforward but tedious so we can get Maple to do it. We get

$$e_{flow} \equiv \int_0^b \int_0^a \langle S_z \rangle \, dx \, dy = \frac{\pi^2 \omega k \left(n^2 a^2 + m^2 b^2\right) B_0^2}{8ab\alpha^2 \mu_0} \tag{18}$$

$$= \frac{\omega abk\omega_{mn}^2}{8c^2\alpha^2\mu_0} B_0^2 \tag{19}$$

$$e_{vol} \equiv \int_0^b \int_0^a \langle u_{em} \rangle \, dx \, dy = \frac{\pi^2 \left(n^2 a^2 + m^2 b^2\right) B_0^2}{16ab\alpha^2\mu_0} \left(\frac{\omega^2}{c^2} + k^2\right) + \frac{abB_0^2}{16\mu_0} \tag{20}$$

$$= \frac{abB_0^2}{16\mu_0} \left[\frac{\omega_{mn}^2}{\alpha^2 c^2}\left(\frac{\omega^2}{c^2} + k^2\right) + 1\right] \tag{21}$$

using 6.

The first integral gives the rate at which energy is flowing along the z direction, while the second integral gives the total energy in a volume of unit length with the cross sectional area of the wave guide. The speed v_E of the energy flow is the rate at which energy flows past a certain point (given by the first integral) divided by the amount of energy in a unit length (the second integral). That is

$$v_E = \frac{e_{flow}}{e_{vol}} \tag{22}$$

$$= \frac{2\omega k\omega_{mn}^2}{c^2\alpha^2} \left[\frac{\omega_{mn}^2}{\alpha^2 c^2}\left(\frac{\omega^2}{c^2} + k^2\right) + 1\right]^{-1} \tag{23}$$

It is actually easier to work out $1/v_E$ first. We get

$$\frac{1}{v_E} = \frac{\frac{\omega^2}{c^2} + k^2}{2\omega k} + \frac{c^2\alpha^2}{2\omega k\omega_{mn}^2} \tag{24}$$

We can eliminate k using 5:

$$\alpha = \frac{\omega^2}{c^2} - k^2 \tag{25}$$

$$= \frac{\omega_{mn}^2}{c^2} \tag{26}$$

$$\frac{1}{v_E} = \frac{2\omega^2 - \omega_{mn}^2}{2\omega c\sqrt{\omega^2 - \omega_{mn}^2}} + \frac{\omega_{mn}^2}{2\omega c\sqrt{\omega^2 - \omega_{mn}^2}} \tag{27}$$

$$= \frac{\omega}{c\sqrt{\omega^2 - \omega_{mn}^2}} \tag{28}$$

Therefore, the speed of the energy flow is

$$v_E = c\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} \tag{29}$$

This is less than c since ω_{mn} is the minimum frequency at which the mode TE$_{mn}$ will propagate in the wave guide, so $\omega_{mn} \leq \omega$.

From 5, the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = v_E \tag{30}$$

so we see that the energy propagates at the group velocity.