RECTANGULAR WAVE GUIDE: TRANSVERSE MAGNETIC MODES

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We can work out the theory of a transverse magnetic (TM) wave in a rectangular wave guide of dimensions $a$ in the $x$ direction and $b$ in the $y$ direction, in the same way as for the TE wave. In a TM wave the component $B_z$ parallel to the axis of the wave guide is zero, so we have only a single wave equation to solve.

\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\omega^2}{c^2} - k^2 \right] E_z = 0 \quad (1)
\]

As this is identical to the equation for TE waves, we have the same solution:

\[
E_z = X(x)Y(y) \quad (2)
\]

\[
X(x) = A \sin k_x x + B \cos k_x x \quad (3)
\]

\[
Y(y) = C \sin k_y y + D \cos k_y y \quad (4)
\]

The boundary conditions are different here, however, as we require

\[
E_1 \parallel - E_2 \parallel = 0 \quad (5)
\]

If the wave guide is a perfect conductor, then $E = 0$ inside it, so $E \parallel = 0$ at all boundaries of the guide. In particular, $E_z = 0$ for $x = 0$, $a$ and $y = 0$, $b$. This means $B = D = 0$, so

\[
E_z(x, y) = E_0 \sin k_x x \sin k_y y \quad (6)
\]

and

\[
k_x = \frac{m \pi}{a} \quad (7)
\]

\[
k_y = \frac{n \pi}{b} \quad (8)
\]
Because $E_z$ uses sines rather than cosines, the integers $m$ and $n$ must both be non-zero in order for the wave to exist at all, so the lowest TM mode is TM$_{11}$.

The wave number has the same form as in the TE case:

$$k = \sqrt{\frac{\omega^2}{c^2} - \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}$$  \hspace{1cm} (9)

$$\equiv \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2}$$  \hspace{1cm} (10)

$$\omega_{mn} = c\pi \sqrt{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}$$  \hspace{1cm} (11)

The phase and group velocities are the same as for TE waves

$$v_p = \frac{\omega}{k}$$  \hspace{1cm} (12)

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}$$  \hspace{1cm} (13)

The ratio of lowest cutoff frequencies is

$$\frac{\omega_{TM}^{11}}{\omega_{TE}^{10}} = \frac{\pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}{\pi c/a} = \frac{1}{b} \sqrt{a^2 + b^2}$$  \hspace{1cm} (14)