RECTANGULAR WAVE GUIDE: TRANSVERSE MAGNETIC MODES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.30.

We can work out the theory of a transverse magnetic (TM) wave in a rectangular wave guide of dimensions a in the x direction and b in the y direction, in the same way as for the TE wave. In a TM wave the component B_z parallel to the axis of the wave guide is zero, so we have only a single wave equation to solve.

$$\left[\partial_{xx}+\partial_{yy}+\frac{\omega^2}{c^2}-k^2\right]E_z = 0 \tag{1}$$

As this is identical to the equation for TE waves, we have the same solution:

$$E_z = X(x)\,Y(y) \tag{2}$$
$$X(x) = A\sin k_x x + B\cos k_x x \tag{3}$$
$$Y(y) = C\sin k_y y + D\cos k_y y \tag{4}$$

The boundary conditions are different here, however, as we require

$$\mathbf{E}_1^{\parallel}-\mathbf{E}_2^{\parallel}=0 \tag{5}$$

If the wave guide is a perfect conductor, then $\mathbf{E}=0$ inside it, so $\mathbf{E}^{\parallel}=0$ at all boundaries of the guide. In particular, $E_z=0$ for $x=0,\ a$ and $y=0,\ b$. This means $B=D=0$, so

$$E_z(x,y)=E_0\sin k_x x\sin k_y y \tag{6}$$

and

$$k_x = \frac{m\pi}{a} \tag{7}$$
$$k_y = \frac{n\pi}{b} \tag{8}$$

Because E_z uses sines rather than cosines, the integers m and n must both be non-zero in order for the wave to exist at all, so the lowest TM mode is TM_{11}.

The wave number has the same form as in the TE case:

$$k = \sqrt{\frac{\omega^2}{c^2} - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)} \tag{9}$$

$$\equiv \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2} \tag{10}$$

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)} \tag{11}$$

The phase and group velocities are the same as for TE waves

$$v_p = \frac{\omega}{k} \tag{12}$$

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} \tag{13}$$

The ratio of lowest cutoff frequencies is

$$\frac{\omega_{11}^{TM}}{\omega_{10}^{TE}} = \frac{\pi c\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}{\pi c/a} = \frac{1}{b}\sqrt{a^2 + b^2} \tag{14}$$