

RECTANGULAR WAVE GUIDE: TRANSVERSE MAGNETIC MODES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.30.

We can work out the theory of a transverse magnetic (TM) wave in a rectangular wave guide of dimensions a in the x direction and b in the y direction, in the same way as for the TE wave. In a TM wave the component B_z parallel to the axis of the wave guide is zero, so we have only a single wave equation to solve.

$$\left[\partial_{xx} + \partial_{yy} + \frac{\omega^2}{c^2} - k^2 \right] E_z = 0 \quad (1)$$

As this is identical to the equation for TE waves, we have the same solution:

$$E_z = X(x)Y(y) \quad (2)$$

$$X(x) = A \sin k_x x + B \cos k_x x \quad (3)$$

$$Y(y) = C \sin k_y y + D \cos k_y y \quad (4)$$

The boundary conditions are different here, however, as we require

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \quad (5)$$

If the wave guide is a perfect conductor, then $\mathbf{E} = 0$ inside it, so $\mathbf{E}^{\parallel} = 0$ at all boundaries of the guide. In particular, $E_z = 0$ for $x = 0, a$ and $y = 0, b$. This means $B = D = 0$, so

$$E_z(x, y) = E_0 \sin k_x x \sin k_y y \quad (6)$$

and

$$k_x = \frac{m\pi}{a} \quad (7)$$

$$k_y = \frac{n\pi}{b} \quad (8)$$

Because E_z uses sines rather than cosines, the integers m and n must both be non-zero in order for the wave to exist at all, so the lowest TM mode is TM_{11} .

The wave number has the same form as in the TE case:

$$k = \sqrt{\frac{\omega^2}{c^2} - \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} \quad (9)$$

$$\equiv \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2} \quad (10)$$

$$\omega_{mn} = c\pi \sqrt{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} \quad (11)$$

The phase and group velocities are the same as for TE waves

$$v_p = \frac{\omega}{k} \quad (12)$$

$$v_g = \frac{d\omega}{dk} = c \sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}} \quad (13)$$

The ratio of lowest cutoff frequencies is

$$\frac{\omega_{11}^{TM}}{\omega_{10}^{TE}} = \frac{\pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}{\pi c / a} = \frac{1}{b} \sqrt{a^2 + b^2} \quad (14)$$