

COAXIAL WAVE GUIDES: TEM MODE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.31.

Although purely transverse electromagnetic (TEM) waves can't exist within a hollow wave guide, it is possible to have TEM waves in a coaxial transmission line. To see this, we start with Maxwell's equations in the form

$$\partial_y E_z - ikE_y = i\omega B_x \quad (1)$$

$$-\partial_x E_z + ikE_x = i\omega B_y \quad (2)$$

$$\partial_x E_y - \partial_y E_x = i\omega B_z \quad (3)$$

$$\partial_y B_z - ikB_y = -i\frac{\omega}{c^2} E_x \quad (4)$$

$$-\partial_x B_z + ikB_x = -i\frac{\omega}{c^2} E_y \quad (5)$$

$$\partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2} E_z \quad (6)$$

By setting $B_z = E_z = 0$ we see from 1 and 5 that

$$B_x = -\frac{k}{\omega} E_y \quad (7)$$

$$-\frac{k^2}{\omega} E_y = -\frac{\omega}{c^2} E_y \quad (8)$$

$$k = \frac{\omega}{c} \quad (9)$$

Substituting this into 1 and 2 we get

$$E_y = -cB_x \quad (10)$$

$$E_x = cB_y \quad (11)$$

from which we see that

$$\mathbf{E} \cdot \mathbf{B} = [cB_y, -cB_x, 0] \cdot [B_x, B_y, 0] = 0 \quad (12)$$

so the electric and magnetic fields are perpendicular to each other.

From 3, 6 and the other two Maxwell equations for vacuum: $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ we get

$$\partial_x E_y - \partial_y E_x = 0 \quad (13)$$

$$\partial_x B_y - \partial_y B_x = 0 \quad (14)$$

$$\partial_x E_x + \partial_y E_y = 0 \quad (15)$$

$$\partial_x B_x + \partial_y B_y = 0 \quad (16)$$

Since the components of E and B don't depend on z [remember the z dependence is contained in the complex exponential in the form $\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y) e^{i(kz - \omega t)}$ and $\tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y) e^{i(kz - \omega t)}$] the first equation is equivalent to saying $\nabla \times \tilde{\mathbf{E}}_0 = 0$ and the second to $\nabla \times \tilde{\mathbf{B}}_0 = 0$. Together with $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, these are Maxwell's equations for static fields [$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{B}}{\partial t} = 0$] in empty space (no free charge or current). In a cylindrical coaxial cable with an inner cylinder of radius a and an outer cylinder of radius b , the magnetic field is, for $a < r < b$

$$\mathbf{B}_0 = \frac{\mu_0 I}{2\pi r} \hat{\phi} \quad (17)$$

where I is the steady current in the inner cylinder. The electric field due to an infinite line of charge is

$$\mathbf{E}_0 = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r} \quad (18)$$

where λ is the linear charge density. These are formal solutions for the case of cylindrical symmetry; the important thing is that the fields have the forms

$$\mathbf{B}_0 = \frac{A}{cr} \hat{\phi} \quad (19)$$

$$\mathbf{E}_0 = \frac{A}{r} \hat{r} \quad (20)$$

for some constant A .

Plugging these into the full formulas for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ and taking the real part, we get

$$\mathbf{E} = \frac{A \cos(kz - \omega t)}{r} \hat{\mathbf{r}} \quad (21)$$

$$\mathbf{B} = \frac{A \cos(kz - \omega t)}{cr} \hat{\boldsymbol{\phi}} \quad (22)$$

These equations satisfy Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{1}{r} \frac{\partial}{\partial r} (rE_r) = 0 \quad (23)$$

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial B_\phi}{\partial \phi} = 0 \quad (24)$$

$$\nabla \times \mathbf{E} = \frac{\partial E_r}{\partial z} \hat{\boldsymbol{\phi}} \quad (25)$$

$$= -k \frac{A \sin(kz - \omega t)}{r} \hat{\boldsymbol{\phi}} \quad (26)$$

$$= -\frac{\omega A \sin(kz - \omega t)}{c} \frac{1}{r} \hat{\boldsymbol{\phi}} \quad (27)$$

$$= -\frac{\partial \mathbf{B}}{\partial t} \quad (28)$$

$$\nabla \times \mathbf{B} = -\frac{\partial B_\phi}{\partial z} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial}{\partial r} (rB_\phi) \hat{\mathbf{z}} \quad (29)$$

$$= k \frac{A \sin(kz - \omega t)}{cr} \hat{\mathbf{r}} \quad (30)$$

$$= \frac{\omega A \sin(kz - \omega t)}{c^2 r} \hat{\mathbf{r}} \quad (31)$$

$$= \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (32)$$

The boundary conditions are

$$B_1^\perp - B_2^\perp = 0 \quad (33)$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \quad (34)$$

and since \mathbf{B} is circumferential, its component normal to the cylinders is zero, and since \mathbf{E} is radial, its parallel component is zero, so the boundary conditions are satisfied.

By comparison with 17 and 18 we can write the analogs for the full fields

$$\mathbf{B} = \frac{\mu_0 I(z, t)}{2\pi r} \hat{\phi} \quad (35)$$

$$\mathbf{E} = \frac{\lambda(z, t)}{2\pi\epsilon_0 r} \hat{\mathbf{r}} \quad (36)$$

From this we can read off the current and charge density on the inner cylinder.

$$I(z, t) = \frac{2\pi A \cos(kz - \omega t)}{\mu_0 c} \quad (37)$$

$$\lambda(z, t) = 2\pi\epsilon_0 A \cos(kz - \omega t) \quad (38)$$