

COAXIAL WAVE GUIDES: TEM MODE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.31.

Although purely transverse electromagnetic (TEM) waves can't exist within a hollow wave guide, it is possible to have TEM waves in a coaxial transmission line. To see this, we start with Maxwell's equations in the form

$$(0.1) \quad \partial_y E_z - ikE_y = i\omega B_x$$

$$(0.2) \quad -\partial_x E_z + ikE_x = i\omega B_y$$

$$(0.3) \quad \partial_x E_y - \partial_y E_x = i\omega B_z$$

$$(0.4) \quad \partial_y B_z - ikB_y = -i\frac{\omega}{c^2}E_x$$

$$(0.5) \quad -\partial_x B_z + ikB_x = -i\frac{\omega}{c^2}E_y$$

$$(0.6) \quad \partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2}E_z$$

By setting $B_z = E_z = 0$ we see from 0.1 and 0.5 that

$$(0.7) \quad B_x = -\frac{k}{\omega}E_y$$

$$(0.8) \quad -\frac{k^2}{\omega}E_y = -\frac{\omega}{c^2}E_y$$

$$(0.9) \quad k = \frac{\omega}{c}$$

Substituting this into 0.1 and 0.2 we get

$$(0.10) \quad E_y = -cB_x$$

$$(0.11) \quad E_x = cB_y$$

from which we see that

$$(0.12) \quad \mathbf{E} \cdot \mathbf{B} = [cB_y, -cB_x, 0] \cdot [B_x, B_y, 0] = 0$$

so the electric and magnetic fields are perpendicular to each other.

From 0.3, 0.6 and the other two Maxwell equations for vacuum: $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ we get

$$(0.13) \quad \partial_x E_y - \partial_y E_x = 0$$

$$(0.14) \quad \partial_x B_y - \partial_y B_x = 0$$

$$(0.15) \quad \partial_x E_x + \partial_y E_y = 0$$

$$(0.16) \quad \partial_x B_x + \partial_y B_y = 0$$

Since the components of E and B don't depend on z [remember the z dependence is contained in the complex exponential in the form $\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y) e^{i(kz - \omega t)}$ and $\tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y) e^{i(kz - \omega t)}$] the first equation is equivalent to saying $\nabla \times \tilde{\mathbf{E}}_0 = 0$ and the second to $\nabla \times \tilde{\mathbf{B}}_0 = 0$. Together with $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, these are Maxwell's equations for static fields [$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{B}}{\partial t} = 0$] in empty space (no free charge or current). In a cylindrical coaxial cable with an inner cylinder of radius a and an outer cylinder of radius b , the magnetic field is, for $a < r < b$

$$(0.17) \quad \mathbf{B}_0 = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

where I is the steady current in the inner cylinder. The electric field due to an infinite line of charge is

$$(0.18) \quad \mathbf{E}_0 = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

where λ is the linear charge density. These are formal solutions for the case of cylindrical symmetry; the important thing is that the fields have the forms

$$(0.19) \quad \mathbf{B}_0 = \frac{A}{cr} \hat{\phi}$$

$$(0.20) \quad \mathbf{E}_0 = \frac{A}{r} \hat{r}$$

for some constant A .

Plugging these into the full formulas for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ and taking the real part, we get

$$(0.21) \quad \mathbf{E} = \frac{A \cos(kz - \omega t)}{r} \hat{\mathbf{r}}$$

$$(0.22) \quad \mathbf{B} = \frac{A \cos(kz - \omega t)}{cr} \hat{\boldsymbol{\phi}}$$

These equations satisfy Maxwell's equations:

$$(0.23) \quad \nabla \cdot \mathbf{E} = \frac{1}{r} \frac{\partial}{\partial r} (rE_r) = 0$$

$$(0.24) \quad \nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial B_\phi}{\partial \phi} = 0$$

$$(0.25) \quad \nabla \times \mathbf{E} = \frac{\partial E_r}{\partial z} \hat{\boldsymbol{\phi}}$$

$$(0.26) \quad = -k \frac{A \sin(kz - \omega t)}{r} \hat{\boldsymbol{\phi}}$$

$$(0.27) \quad = -\frac{\omega A \sin(kz - \omega t)}{c} \frac{1}{r} \hat{\boldsymbol{\phi}}$$

$$(0.28) \quad = -\frac{\partial \mathbf{B}}{\partial t}$$

$$(0.29) \quad \nabla \times \mathbf{B} = -\frac{\partial B_\phi}{\partial z} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial}{\partial r} (rB_\phi) \hat{\mathbf{z}}$$

$$(0.30) \quad = k \frac{A \sin(kz - \omega t)}{cr} \hat{\mathbf{r}}$$

$$(0.31) \quad = \frac{\omega A \sin(kz - \omega t)}{c^2 r} \hat{\mathbf{r}}$$

$$(0.32) \quad = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

The boundary conditions are

$$(0.33) \quad B_1^\perp - B_2^\perp = 0$$

$$(0.34) \quad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0$$

and since \mathbf{B} is circumferential, its component normal to the cylinders is zero, and since \mathbf{E} is radial, its parallel component is zero, so the boundary conditions are satisfied.

By comparison with 0.17 and 0.18 we can write the analogs for the full fields

$$(0.35) \quad \mathbf{B} = \frac{\mu_0 I(z, t)}{2\pi r} \hat{\phi}$$

$$(0.36) \quad \mathbf{E} = \frac{\lambda(z, t)}{2\pi\epsilon_0 r} \hat{r}$$

From this we can read off the current and charge density on the inner cylinder.

$$(0.37) \quad I(z, t) = \frac{2\pi A \cos(kz - \omega t)}{\mu_0 c}$$

$$(0.38) \quad \lambda(z, t) = 2\pi\epsilon_0 A \cos(kz - \omega t)$$