COAXIAL WAVE GUIDES: TEM MODE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.31.

Although purely transverse electromagnetic (TEM) waves can't exist within a hollow wave guide, it is possible to have TEM waves in a coaxial transmission line. To see this, we start with Maxwell's equations in the form

$$\partial_y E_z - ikE_y = i\omega B_x \tag{1}$$
$$-\partial_x E_z + ikE_x = i\omega B_y \tag{2}$$
$$\partial_x E_y - \partial_y E_x = i\omega B_z \tag{3}$$

$$\partial_y B_z - ikB_y = -i\frac{\omega}{c^2}E_x \tag{4}$$
$$-\partial_x B_z + ikB_x = -i\frac{\omega}{c^2}E_y \tag{5}$$
$$\partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2}E_z \tag{6}$$

By setting $B_z = E_z = 0$ we see from 1 and 5 that

$$B_x = -\frac{k}{\omega}E_y \tag{7}$$
$$-\frac{k^2}{\omega}E_y = -\frac{\omega}{c^2}E_y \tag{8}$$
$$k = \frac{\omega}{c} \tag{9}$$

Substituting this into 1 and 2 we get

$$E_y = -cB_x \tag{10}$$
$$E_x = cB_y \tag{11}$$

from which we see that

$$\mathbf{E}\cdot\mathbf{B} = [cB_y, -cB_x, 0]\cdot[B_x, B_y, 0] = 0 \tag{12}$$

so the electric and magnetic fields are perpendicular to each other.

From 3, 6 and the other two Maxwell equations for vacuum: $\nabla\cdot\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$ we get

$$\begin{aligned}\partial_x E_y-\partial_y E_x &= 0 &(13)\\ \partial_x B_y-\partial_y B_x &= 0 &(14)\\ \partial_x E_x+\partial_y E_y &= 0 &(15)\\ \partial_x B_x+\partial_y B_y &= 0 &(16)\end{aligned}$$

Since the components of E and B don't depend on z [remember the z dependence is contained in the complex exponential in the form $\tilde{\mathbf{E}}(x,y,z,t)=\tilde{\mathbf{E}}_0(x,y)\,e^{i(kz-\omega t)}$ and $\tilde{\mathbf{B}}(x,y,z,t)=\tilde{\mathbf{B}}_0(x,y)\,e^{i(kz-\omega t)}$] the first equation is equivalent to saying $\nabla\times\tilde{\mathbf{E}}_0=0$ and the second to $\nabla\times\tilde{\mathbf{B}}_0=0$. Together with $\nabla\cdot\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$, these are Maxwell's equations for static fields $[\frac{\partial\mathbf{E}}{\partial t}=\frac{\partial\mathbf{B}}{\partial t}=0]$ in empty space (no free charge or current). In a cylindrical coaxial cable with an inner cylinder of radius a and an outer cylinder of radius b, the magnetic field is, for $a<r<b$

$$\mathbf{B}_0=\frac{\mu_0 I}{2\pi r}\hat{\phi} \tag{17}$$

where I is the steady current in the inner cylinder. The electric field due to an infinite line of charge is

$$\mathbf{E}_0=\frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}} \tag{18}$$

where λ is the linear charge density. These are formal solutions for the case of cylindrical symmetry; the important thing is that the fields have the forms

$$\begin{aligned}\mathbf{B}_0 &= \frac{A}{cr}\hat{\phi} &(19)\\ \mathbf{E}_0 &= \frac{A}{r}\hat{\mathbf{r}} &(20)\end{aligned}$$

for some constant A.

Plugging these into the full formulas for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ and taking the real part, we get

$$\mathbf{E} = \frac{A\cos(kz-\omega t)}{r}\hat{\mathbf{r}} \tag{21}$$

$$\mathbf{B} = \frac{A\cos(kz-\omega t)}{cr}\hat{\phi} \tag{22}$$

These equations satisfy Maxwell's equations:

$$\nabla\cdot\mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = 0 \tag{23}$$

$$\nabla\cdot\mathbf{B} = \frac{1}{r}\frac{\partial B_\phi}{\partial\phi} = 0 \tag{24}$$

$$\nabla\times\mathbf{E} = \frac{\partial E_r}{\partial z}\hat{\phi} \tag{25}$$

$$= -k\frac{A\sin(kz-\omega t)}{r}\hat{\phi} \tag{26}$$

$$= -\frac{\omega}{c}\frac{A\sin(kz-\omega t)}{r}\hat{\phi} \tag{27}$$

$$= -\frac{\partial\mathbf{B}}{\partial t} \tag{28}$$

$$\nabla\times\mathbf{B} = -\frac{\partial B_\phi}{\partial z}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\phi\right)\hat{\mathbf{z}} \tag{29}$$

$$= k\frac{A\sin(kz-\omega t)}{cr}\hat{\mathbf{r}} \tag{30}$$

$$= \frac{\omega A\sin(kz-\omega t)}{c^2 r}\hat{\mathbf{r}} \tag{31}$$

$$= \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{32}$$

The boundary conditions are

$$B_1^\perp - B_2^\perp = 0 \tag{33}$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \tag{34}$$

and since $\mathbf{B}$ is circumferential, its component normal to the cylinders is zero, and since $\mathbf{E}$ is radial, its parallel component is zero, so the boundary conditions are satisfied.

By comparison with 17 and 18 we can write the analogs for the full fields

$$\mathbf{B} = \frac{\mu_0 I(z,t)}{2\pi r}\hat{\phi} \tag{35}$$

$$\mathbf{E} = \frac{\lambda(z,t)}{2\pi\epsilon_0 r}\hat{\mathbf{r}} \tag{36}$$

From this we can read off the current and charge density on the inner cylinder.

$$I(z,t) = \frac{2\pi A\cos(kz-\omega t)}{\mu_0 c} \tag{37}$$

$$\lambda(z,t) = 2\pi\epsilon_0 A\cos(kz-\omega t) \tag{38}$$